

# Anatomy of the chase

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## ABSTRACT

A lot of research activity has recently taken place around the chase procedure, due to its usefulness in data integration, data exchange, query optimization, peer data exchange and data correspondence, to mention a few. As the chase has been investigated and further developed by a number of research groups and authors, many variants of the chase have emerged and associated results obtained. Due to the heterogeneous nature of the area it is frequently difficult to verify the scope of each result. In this paper we take closer look at recent developments, and provide additional results. Our analysis allows us create a taxonomy of the chase variations and the properties they satisfy.

Two of the most central problems regarding the chase is termination, and discovery of restricted classes of sets of dependencies that guarantee termination of the chase. The search for the restricted classes has been motivated by a fairly recent result that shows that it is undecidable to determine whether the chase with a given dependency set will terminate on a given instance. There is a small dissonance here, since the quest has been for classes of sets of dependencies guaranteeing termination of the chase on *all* instances, even though the latter problem was not known to be undecidable. We resolve the dissonance in this paper by showing that determining whether the chase with a given set of dependencies terminates on all instances is coRE-complete. Our reduction also gives us the aforementioned instance-dependent RE-completeness result as a byproduct. For one of the restricted classes, the *stratified* sets dependencies, we provide new complexity results for the problem of testing whether a given set of dependencies belongs to it. These results rectify some previous claims that have occurred in the literature.

## Categories and Subject Descriptors

H.2.5 [Heterogeneous Databases]: Data translation

## General Terms

Algorithms, Theory

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## Keywords

Chase, Data Exchange, Data Repair, Incomplete databases, Undecidability Complexity

## 1. INTRODUCTION

The chase procedure was initially developed for testing logical implication between sets of dependencies [3], for determining equivalence of database instances known to satisfy a given set of dependencies [21, 16], and for determining query equivalence under database constraints [2]. Recently the chase has experienced a revival due to its application in data integration, data exchange, data repair, query optimization, ontologies and data correspondence. In this paper we will focus on constraints in the form of embedded dependencies [8] specified by sets of tuple generating dependencies (tgds). A tgd is a first order formula of the form

$$\forall \bar{x}, \bar{y} (\alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z})),$$

where  $\alpha$  and  $\beta$  are conjunctions of relational atoms, and  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$  are sequences of variables. We refer to  $\alpha$  as the body and  $\beta$  as the head of the dependency. Sometimes, for simplicity, the tgd is written as  $\alpha \rightarrow \beta$ . Intuitively the chase procedure repeatedly applies a series of chase steps to database instances that violate some dependency. Each such chase step takes a tgd that is not satisfied by the instance, a set of tuples that witness the violation, and adds new tuples to the database instance so that the resulting instance does satisfy the tgd with respect to those witnessing tuples.

Given an instance  $I$  and a set of tgd's  $\Sigma$ , a *model* of  $I$  and  $\Sigma$  is a database instance  $J$  such that there is a homomorphism from  $I$  to  $J$ , and  $J$  satisfies  $\Sigma$ . A *universal model* of  $I$  and  $\Sigma$  is a *finite* model of  $I$  and  $\Sigma$  that has a homomorphism into *every* model of  $I$  and  $\Sigma$ . It was shown in [9, 6] that the chase computes a universal model of  $I$  and  $\Sigma$ , whenever  $I$  and  $\Sigma$  has one. In case  $I$  and  $\Sigma$  does not have a universal model the chase

doesn't terminate (in this case it actually converges at a countably infinite model).

As the research on the chase has progressed several variations of the chase have evolved. As a consequence it has become difficult to determine the scope of the results obtained. We scrutinize the four most important chase variations, both deterministic and non-deterministic. We will check for each of these chase variations the data and combined complexity of testing if the chase step is applicable for a given instance and  $\text{tgd}$ . It didn't come as a surprise to find out that the oblivious and semi-oblivious chase variation share the same complexity, but the standard chase has a slightly higher complexity. The table below shows the data and combined complexity for the following problem: given an instance with  $n$  tuples and a  $\text{tgd}$  ( $\alpha \rightarrow \beta$ ), is the standard/oblivious chase step applicable?

Chase	Data Complexity	Combined Complexity
standard	$O(n^{ \alpha + \beta })$	$\Sigma_2^P$ -complete
oblivious	$O(n^{ \alpha })$	NP-complete

Thus, at a first look the oblivious and semi-oblivious chase procedures will be a more appropriate choice when it comes to a practical implementation. Still, as we will show, the lower complexity comes with a price, that is the higher the complexity for a chase variation the more "likely" is the chase process to terminate for a given instance and set of dependencies. Thus, the core chase, that not only applies in parallel all standard chase steps but it also computes the core of the resulted instance, has the highest complexity of the chase step. On the other hand from [6] we know that the core chase is complete in finding universal models meaning that if any of the chase variations terminates for some input, then the core chase terminates as well. We next compare the semi-oblivious and standard chase when it comes to the termination problem. With this we show that the standard and semi-oblivious chase are not distinguishable for the most classes of dependencies developed to ensure the standard chase termination. Furthermore, we show that the number of semi-oblivious chase steps needed to terminate remains the same as for the standard chase, namely polynomial. This raises the following question: What makes a class of dependency sets that terminate for all input instances under the standard chase to terminate for the semi-oblivious chase as well? We answer this question by giving a sufficient syntactical condition for classes of dependency sets that ensures termination on all instances for the standard chase to also guarantee termination for the semi-oblivious chase. As we will see most of the known classes of dependencies build to ensure the standard chase termination on all instances are

actually guaranteeing termination for the less expensive semi-oblivious chase variation.

It has been known for some time [6, 4, 17] that it is undecidable to determine if the chase with a given set of  $\text{tgd}$ 's terminates on a given instance. This has spurred a quest for restricted classes of  $\text{tgd}$ 's guaranteeing termination. Interestingly, these classes all guarantee *uniform* termination, that termination on *all* instances. This, even though it was only known that the problem is undecidable for a given instance. We remediate the situation by proving that (perhaps not too surprisingly) the uniform version of the termination problem is undecidable as well, and show that it is not recursively enumerable. We show that the determining whether the core chase with a given set of dependencies terminates on all instances is **coRE**-complete. We achieve this using a reduction from the uniform termination problem for word-rewriting systems (semi-Thue systems). As a byproduct we obtain the result from [6] showing that testing if the core chase terminates for a given instance and a given set of dependencies is **RE**-complete. We will show also that the same complexity result holds for testing whether the standard chase with a set of dependencies is terminating on at least one execution branch. Next we will show that by using a single denial constraint (a "headless"  $\text{tgd}$ ) in our reduction the same complexity result holds also for the standard chase termination on all instances on all execution branches. It remains an open problem if this holds without denial constraints.

Many of the restricted classes guaranteeing termination rely on the notion of a set  $\Sigma$  of dependencies being *stratified*. Stratification involves two conditions, one determining a partial order between  $\text{tgd}$ 's in  $\Sigma$ , and the other on  $\Sigma$  as a whole. It has been claimed that testing the partial order between  $\text{tgd}$ 's is in **NP** [6]. We show that this cannot be the case (unless **NP=coNP**), by proving that the problem is at least **coNP**-hard. We also prove a  $\Delta_2^P$  upper bound for the problem. Finding matching upper and lower bounds remains an open problem.

### Paper outline

The next section contains the preliminaries and describes the chase procedure and its variation. Section 3 deals with the complexity of testing if for an instance and a dependency there exists an "applicable" chase step. Section 4 deals with problems related to the chase termination. We define termination classes for each of the chase variations and then determine the relationship between these classes. Section 4 also contains our main result, namely, that it is **coRE**-complete to test if the chase variations with a given set of dependencies terminate on all instances. This result is obtained via a reduction from the uniform termination problem

for word-rewriting systems. In Section 5 we review the main restricted classes that ensure termination on all instances, and relate them to different chase variations and their termination classes. Finally, in Section 6 we provide complexity results related to the membership problem for the stratification based classes of dependencies that ensure the standard chase termination. Conclusions and further work are drawn in the last section. Proofs not given in the paper are included in an Appendix.

## 2. PRELIMINARIES

For basic definitions and concepts we refer to [1]. We will consider the complexity classes PTIME, NP, coNP, DP, RE, coRE, and the first few levels of the polynomial hierarchy. For the definitions of these classes we refer to [23].

We start with some preliminary notions. We will use the symbol  $\subseteq$  for the subset relation, and  $\subset$  for proper subset. A function  $f$  with a finite set  $\{x_1, \dots, x_n\}$  as domain, and where  $f(x_i) = a_i$ , will be described as  $\{x_1/a_1, \dots, x_n/a_n\}$ . The reader is cautioned that the symbol  $\rightarrow$  will be overloaded; the meaning should however be clear from the context.

**Relational schemas and instances.** A *relational schema* is a finite set  $\mathbf{R} = \{R_1, \dots, R_n\}$  of relational symbols  $R_i$ , each with an associated positive integer *arity*( $R_i$ ). Let **Cons** be a countably infinite set of constants, usually denoted  $a, b, c, \dots$ , possibly subscripted, and let **Nulls** be a countably infinite set of nulls denoted  $x, y_1, y_2, \dots$ . A *relational instance* over a schema  $\mathbf{R}$  is a function that associates for each relational symbol  $R \in \mathbf{R}$  a finite subset  $R^I$  of  $(\mathbf{Cons} \cup \mathbf{Nulls})^{\text{arity}(R)}$ .

For notational convenience we shall frequently identify an instance  $I$  with the set  $\{R(\bar{a}) : (\bar{a}) \in R^I, R \in \mathbf{R}\}$  of atoms, assuming appropriate lengths of the sequence  $\bar{a}$  for each  $R \in \mathbf{R}$ . By the same convenience, the atoms  $R(a_1 \dots, a_k)$  will be called *tuples* of relation  $R^I$  and denoted  $t, t_1, t_2, \dots$ . By  $\text{dom}(I)$  we mean the set of all constants and nulls occurring in the instance  $I$ , and by  $|I|$  we mean the number of tuples in  $I$ .

**Homomorphisms.** Let  $I$  and  $J$  be instances, and  $h$  a mapping from  $\text{dom}(I)$  to  $\text{dom}(J)$  that is the identity on the constants. We extend  $h$  to tuples  $(\bar{a}) = (a_1, \dots, a_k)$  by  $h(a_1, \dots, a_k) = (h(a_1), \dots, h(a_k))$ . By our notational convenience we can thus write  $h(\bar{a})$  as  $h(R(\bar{a}))$ , when  $(\bar{a}) \in R^I$ . We extend  $h$  to instances by  $h(I) = \{h(t) : t \in I\}$ . If  $h(I) \subseteq J$  we say that  $h$  is a *homomorphism* from  $I$  to  $J$ . If  $h(I) \subseteq I$ , we say that  $h$  is an *endomorphism*, and if  $h$  also is idempotent,  $h$  is called a *retraction*. If  $h(I) \subseteq J$ , and the mapping  $h$  is a

bijection, and if also  $h^{-1}(J) = I$ , the two instances are *isomorphic*.

A subset  $I'$  of  $I$  is said to be a *core* of  $I$ , if there is an endomorphism  $h$ , such that  $h(I) \subseteq I'$ , and there is no endomorphism  $g$  such that  $g(I') \subset I'$ . It is well known that all cores of an instance  $I$  are isomorphic, so for our purposes we can consider the core unique, and denote it  $\text{core}(I)$ .

**Tuple generating dependencies.** A *tuple generating dependency* (*tgd*) is a first order formula of the form

$$\forall \bar{x}, \bar{y} \left( \alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z}) \right),$$

where  $\alpha$  and  $\beta$  are conjunctions of relational atoms, and  $\bar{x}, \bar{y}$  and  $\bar{z}$  are sequences of variables. We assume that the variables occurring in *tgd*'s come from a countably infinite set **Vars** disjoint from **Nulls**. We also allow constants in the *tgd*'s. In the formula we call  $\alpha$  the *body* of the *tgd*. Similarly we refer to  $\beta$  as the *head* of the *tgd*. If there are no existentially quantified variables the dependency is said to be *full*.

When  $\alpha$  is the body of a *tgd* and  $h$  a mapping from the set  $\mathbf{Vars} \cup \mathbf{Const}$  to  $\mathbf{Nulls} \cup \mathbf{Const}$  that is identity on constants, we shall conveniently regard the set of atoms in  $\alpha$  as an instance  $I_\alpha$ , and write  $h(\alpha)$  for the set  $h(I_\alpha)$ . Then  $h$  is a homomorphism from  $\alpha$  to an instance  $I$ , if  $h(\alpha) \subseteq I$ .

Frequently, we omit the universal quantifiers in *tgd* formulas. Also, when the variables and constants are not relevant in the context, we denote a tuple generating dependency  $\alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z})$  simply as  $\alpha \rightarrow \beta$ .

Let  $\xi = \alpha \rightarrow \beta$  be a tuple generating dependency, and  $I$  be an instance. Then we say that  $I$  *satisfies*  $\xi$ , if  $I \models \xi$  in the standard model theoretic sense, or equivalently, if for every homomorphism  $h$ , such that  $h(\alpha) \subseteq I$ , there is an extension  $h'$  of  $h$ , such that  $h'(\beta) \subseteq I$ .

**The Chase.** Let  $\Sigma$  be a (finite) set of *tgd*'s and  $I$  an instance. A *trigger* for the set  $\Sigma$  on  $I$  is a pair  $(\xi, h)$ , where  $\xi = \alpha \rightarrow \beta \in \Sigma$ , and  $h$  is a homomorphism such that  $h(\alpha) \subseteq I$ . If, in addition, there is no extension  $h'$  of  $h$ , such that  $h'(\beta) \subseteq I$ , we say that the trigger  $(\xi, h)$  is *active* on  $I$ .

Let  $(\xi, h)$  be a trigger for  $\Sigma$  on  $I$ . To *fire* the trigger means transforming  $I$  into the instance  $J = I \cup \{h'(\beta)\}$ , where  $h'$  is a *distinct extension* of  $h$ , i.e. an extension of  $h$  that assigns new fresh nulls to the existential variables in  $\beta$ . By “new fresh” we mean the next unused element in some fixed enumeration of the nulls. We denote this transformation  $I \xrightarrow{(\xi, h')} J$ , or just  $I \rightarrow J$ , if the particular trigger is irrelevant or understood from the context.

A sequence  $I_0, I_1, I_2 \dots$  of instances (finite or infinite) is said to be a *chase sequence with  $\Sigma$  originating from  $I_0$* , if  $I_i \rightarrow I_{i+1}$  for all  $i = 0, 1, 2, \dots$ . At each step there

can naturally be several triggers to choose from, so in general there will be several chase sequences originating from  $I_0$  for any given set  $\Sigma$  of  $\text{tgds}$ . If for some  $i$  that is there are no more triggers to be fired for  $I_i$ , we say that the sequence *terminates*. Otherwise the sequence is infinite.

In summary, the chase process can be seen as a tree rooted at  $I_0$ , and with the individual chase sequences as branches. From an algorithmic point of view the choice of the next trigger to fire is essential. Based on this, the following variations of the chase process have been considered in the literature. (for a comprehensive review of different chase variation see [22]).

1. *The standard chase* [9]. The next trigger is chosen nondeterministically from the subset of current triggers that are active.
2. *The oblivious chase* [4]. The next trigger is chosen nondeterministically from the set of all current triggers, active or not, but each trigger is fired only once in a chase sequence.
3. *The semi-oblivious chase* [17]. Let  $\xi$  be a  $\text{tgd}$   $\alpha(\bar{x}, \bar{y}) \rightarrow \beta(\bar{x}, \bar{z})$ . Then triggers  $(\xi, h)$  and  $(\xi, g)$  are considered equivalent if  $h(\bar{x}) = g(\bar{x})$ . The semi-oblivious chase works as the oblivious one, except that exactly one trigger from each equivalence class is fired in a branch.
4. *The core chase* [6]. At each step, all currently active triggers are fired in parallel, and then the core of the union of the resulting instances is computed before the next step. Note that this makes the chase process deterministic.

The three first variations are all nondeterministic, but differ in which triggers they fire. Also we consider all chase procedures to be *fair*, meaning that in any infinite chase sequence, if a trigger is applicable at some chase step  $i$ , then there exists an integer  $j \geq i$  such that the trigger is fired at step  $j$ .

To illustrate the difference between these chase variations, consider dependency set  $\Sigma = \{\xi\}$ , where  $\xi$  is  $\text{tgd}$   $R(x, y) \rightarrow \exists z S(x, z)$ , and instance:

$$\frac{I_0}{\begin{array}{l} R(a, b) \\ R(a, c) \\ S(a, d) \end{array}}$$

There are two triggers for the set  $\Sigma$  on instance  $I_0$ , namely  $(\xi, \{x/a, y/b\})$  and  $(\xi, \{x/a, y/c\})$ . Since  $I_0 \models \xi$  neither of the triggers is active, so the standard chase will terminate at  $I_0$ . In contrast, both the oblivious and semi-oblivious chase will fire the first trigger, resulting

in instance  $I_1 = I_0 \cup \{S(a, z_1)\}$ . The semi-oblivious chase will terminate at this point, while the oblivious chase will fire the second trigger, and then terminate in  $I_2 = I_1 \cup \{S(a, z_2)\}$ . The core chase in this case will terminate also with  $I_0$ .

### 3. COMPLEXITY OF THE CHASE STEP

Algorithmically, there are two problems to consider. For knowing when to terminate the chase, we need to determine whether for a given instance  $I$  and  $\text{tgd}$   $\xi$  there exists a homomorphism  $h$  such that  $(\xi, h)$  is trigger on  $I$ . This pertains to the oblivious and semi-oblivious variations. The second problem pertains to the standard and core chase: given an instance  $I$  and a  $\text{tgd}$   $\xi$ , is there a homomorphism  $h$ , such that  $(\xi, h)$  is an *active* trigger on  $I$ . We call these problems the *trigger existence* problem, and the *active trigger existence* problem, respectively. The *data complexity* of these problems considers  $\xi$  fixed, and in the *combined complexity* both  $I$  and  $\xi$  are part of the input. The following theorem gives the combined and data complexities of the two problems.

**THEOREM 1.** *Let  $\xi$  be a  $\text{tgd}$  and  $I$  an instance. Then*

1. *For a fixed  $\xi$ , testing whether there exists a trigger or an active trigger on a given  $I$  is polynomial.*
2. *Testing whether there exists a trigger for a given  $\xi$  on a given  $I$  is NP-complete.*
3. *Testing whether there exists an active trigger for a given  $\xi$  and a given  $I$  is  $\Sigma_2^P$ -complete.*

**Proof:** (Sketch) The polynomial cases can be verified by checking all homomorphisms from the body of the dependency into the instance. For the active trigger problem we also need to consider, for each such homomorphism, if it has an extension that maps the head of the dependency into the instance. These tasks can be carried out in  $O(n^{|\alpha|})$  and  $O(n^{|\alpha|+|\beta|})$  time, respectively.

It is easy to see that the trigger existence problem is NP-complete in combined complexity, as the problem is equivalent to testing whether there exists a homomorphism between two instances (in our case  $\alpha$  and  $I$ ); a problem known to be NP-complete.

For the combined complexity of the active trigger existence problem, we observe that it is in  $\Sigma_2^P$ , since one may guess a homomorphism  $h$  from  $\alpha$  into  $I$ , and then use an NP oracle to verify that there is no extension  $h'$  of  $h$ , such that  $h'(\beta) \subseteq I$ . For the lower bound we will reduce the following problem to the active trigger existence problem [24]. Let  $\phi(\bar{x}, \bar{y})$  be a Boolean formula in 3CNF over the variables in  $\bar{x}$  and  $\bar{y}$ . Is the formula

$$\exists \bar{x} \neg (\exists \bar{y} \phi(\bar{x}, \bar{y}))$$

true? The problem is a variation of the standard  $\exists\forall$ -QBF problem [25].

For the reduction, let  $\phi$  be given. We construct an instance  $I_\phi$  and a tgd  $\xi_\phi$ . The instance  $I_\phi$  is as follows:

$F$			$N$	
1	0	0	0	1
0	1	0	1	0
0	0	1		
1	1	0		
1	0	1		
0	1	1		
1	1	1		

The tgd  $\xi_\phi = \alpha \rightarrow \beta$  is constructed as follows. For each variable  $x \in \bar{x}$  in  $\phi(\bar{x}, \bar{y})$ , the body  $\alpha$  will contain the atom  $N(x, x')$  ( $x'$  is used to represent  $\neg x$ ). The head  $\beta$  is existentially quantified over that set  $\bigcup_{y \in \bar{y}} \{y, y'\}$  of variables. For each conjunct  $C$  of  $\phi$ , we place in  $\beta$  an atom  $F(x, y, z)$ , where  $x, y$  and  $z$  are the variables in  $C$ , with the convention that if variable  $x$  is negated in  $C$ , then  $x'$  is used in the atom. Finally for each  $y \in \bar{y}$ , we place in  $\beta$  the atom  $N(y, y')$ , denoting that  $y$  and  $y'$  should not have the same truth assignment.

Let us now suppose that the formula  $\exists \bar{x} \neg (\exists \bar{y} \phi(\bar{x}, \bar{y}))$  is true. This means that there is a  $\{0, 1\}$ -valuation  $h$  of  $\bar{x}$  such that for any  $\{0, 1\}$  valuation  $h'$  of  $\bar{y}$ , the formula  $\phi(h(\bar{x}), h'(\bar{y}))$  is false. It is easy to see that  $h(\alpha) \subseteq I$ . Also, since  $\phi(h(\bar{x}), h'(\bar{y}))$  is false for any valuation  $h'$ , for each  $h'$  there must be an atom  $F(x, y, z) \in \beta$ , such that  $h' \circ h(F(x, y, z))$  is false, that is either  $h' \circ h(F(x, y, z)) = F(0, 0, 0) \notin I_\phi$  or  $h'$  assigns for some existentially quantified variables non boolean values. Consequently the trigger  $(\xi, h)$  is active on  $I_\phi$ .

For the other direction, suppose that there exists a trigger  $(\xi, h)$  which is active on  $I_\phi$ , i.e.,  $h(\alpha) \subseteq I_\phi$  and  $h'(\beta) \not\subseteq I$ , for any extension  $h'$  of  $h$ . This means that for any such extension  $h'$ , either  $h'$  is not  $\{0, 1\}$ -valuation, or that the atom  $F(0, 0, 0)$  is in  $h'(\beta)$ . Thus the formula  $\exists \bar{x} \neg (\exists \bar{y} \phi(\bar{x}, \bar{y}))$  is true. ■

Note that the trigger existence relates to the oblivious and semi-oblivious chase variations, whereas the active trigger existence relates to the standard and the core chase. This means that the oblivious and semi-oblivious chase variations have the same complexity. This is not the case for the standard and the core chase, because the core chase step applies all active triggers in parallel and also involves the core computation for the resulted instance. In [10] it is shown that computing the core involves a DP-complete decision problem.

## 4. CHASE TERMINATION QUESTIONS

Being able to decide whether a chase should be terminated at a given step in the sequence does not mean that we can decide whether the case ever will terminate.

The latter problem is undecidable in general, as we will see in the second subsection. However, the several chase variations have different termination behavior, so next we introduce some notions that will help to distinguish them.

### 4.1 Termination classes

Let  $\star \in \{\text{std}, \text{obl}, \text{sobl}, \text{core}\}$ , corresponding to the chase variations introduced in Section 2, and let  $\Sigma$  be a set of tgd's. If there exists a terminating  $\star$ -chase sequence with  $\Sigma$  on  $I$ , we say that the  $\star$ -chase terminates for *some* branch on instance  $I$ , and denote this as  $\Sigma \in \text{CT}_{I, \exists}^*$ . Here  $\text{CT}_{I, \exists}^*$  is thus to be understood as the class of all sets of tgd's for which the  $\star$ -chase terminates on some branch on instance  $I$ . Likewise,  $\text{CT}_{I, \forall}^*$  denotes the class of all sets of tgd's for which the  $\star$ -chase with  $\Sigma$  on  $I$  terminates on *all* branches. From the definition of the chase variations it is easy to observe that any trigger applicable by the standard chase step on an instance  $I$  is also applicable by the semi-oblivious and oblivious chase steps on the same instance. Similarly all the triggers applicable by the semi-oblivious chase step on an instance  $I$  is also applicable by the oblivious chase step on instance  $I$ . Note that the converse is not always true. Thus  $\text{CT}_{I, \exists}^{\text{obl}} \subseteq \text{CT}_{I, \exists}^{\text{sobl}} \subseteq \text{CT}_{I, \exists}^{\text{std}}$ . It is also easy to verify that  $\text{CT}_{I, \exists}^* = \text{CT}_{I, \forall}^*$  for  $\star \in \{\text{obl}, \text{sobl}\}$ , that  $\text{CT}_{I, \forall}^{\text{std}} \subseteq \text{CT}_{I, \exists}^{\text{std}}$ , and that  $\text{CT}_{I, \forall}^{\text{sobl}} \subseteq \text{CT}_{I, \forall}^{\text{std}}$ . The following propositions shows that these results can be strengthened by including strict inclusions:

PROPOSITION 1. *For any instance  $I$  we have:*

$$\text{CT}_{I, \forall}^{\text{obl}} = \text{CT}_{I, \exists}^{\text{obl}} \subset \text{CT}_{I, \forall}^{\text{sobl}} = \text{CT}_{I, \exists}^{\text{sobl}} \subset \text{CT}_{I, \forall}^{\text{std}} \subset \text{CT}_{I, \exists}^{\text{std}}.$$

*Proof:* (Sketch) For strict inclusion  $\text{CT}_{I, \exists}^{\text{obl}} \subset \text{CT}_{I, \forall}^{\text{sobl}}$  consider instance  $I = \{R(a)\}$  and  $\Sigma$  containing a single (tautological) dependency  $R(x) \rightarrow \exists y R(y)$ . It is easy to see that  $\Sigma \in \text{CT}_{I, \forall}^{\text{sobl}}$  but  $\Sigma \notin \text{CT}_{I, \exists}^{\text{obl}}$ . For the second strict inclusion  $\text{CT}_{I, \forall}^{\text{sobl}} \subset \text{CT}_{I, \forall}^{\text{std}}$ , consider instance  $I = \{S(a, a)\}$  and  $\Sigma = \{S(x, y) \rightarrow \exists z S(y, z)\}$ . Because  $I \models \Sigma$  it follows that  $\Sigma \in \text{CT}_{I, \forall}^{\text{std}}$ . On the other hand, the semi-oblivious chase will not terminate with  $\Sigma$  on  $I$ , and thus  $\Sigma \notin \text{CT}_{I, \forall}^{\text{sobl}}$ . For the final strict inclusion  $\text{CT}_{I, \forall}^{\text{std}} \subset \text{CT}_{I, \exists}^{\text{std}}$ , let  $I = \{S(a, b), R(a)\}$  and  $\Sigma = \{S(x, y) \rightarrow \exists z S(y, z); R(x) \rightarrow S(x, x)\}$ . It is easy to see that any standard chase sequence that starts by firing the trigger based on the first tgd will not terminate, whereas if we first fire the trigger based on the second tgd, the standard chase will terminate after one step. ■

The next question is whether a  $\star$ -chase terminates on *all* instances for all or for some branches. The corresponding classes of sets of tgd's are denoted  $\text{CT}_{\forall \forall}^*$  and  $\text{CT}_{\forall \exists}^*$ , respectively. Obviously  $\text{CT}_{\forall \forall}^{\text{std}} \subset \text{CT}_{\forall \exists}^{\text{std}}$ .

Similarly to the instance dependent termination classes,  $\text{CT}_{\forall\forall}^{\text{obl}} \subset \text{CT}_{\forall\forall}^{\text{sobl}}$  and  $\text{CT}_{\forall\forall}^{\star} = \text{CT}_{\forall\exists}^{\star}$ , for  $\star \in \{\text{obl}, \text{sobl}\}$ .

Now can relate the oblivious, semi-oblivious and standard chase termination classes as follows:

THEOREM 2.

$$\text{CT}_{\forall\forall}^{\text{obl}} = \text{CT}_{\forall\exists}^{\text{obl}} \subset \text{CT}_{\forall\forall}^{\text{sobl}} = \text{CT}_{\forall\exists}^{\text{sobl}} \subset \text{CT}_{\forall\forall}^{\text{std}} \subset \text{CT}_{\forall\exists}^{\text{std}}.$$

The proof of this theorem is included in the Appendix.

Note that for any  $\star \in \{\text{std}, \text{obl}, \text{sobl}, \text{core}\}$ , and for any non-empty instance  $I$ , we have that  $\text{CT}_{\forall\forall}^{\star} \subset \text{CT}_{I,\forall}^{\star}$  and  $\text{CT}_{\forall\exists}^{\star} \subset \text{CT}_{I,\exists}^{\star}$ .

The termination of the oblivious chase can be related to the termination of the standard chase by using the *enrichment* transformation, introduced in [11]. The enrichment takes a tgd  $\xi = \alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z})$  over schema  $\mathbf{R}$  and converts it into tuple generating dependency  $\hat{\xi} = \alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z}), H(\bar{x}, \bar{y})$ , where  $H$  is a new relational symbol that does not appear in  $\mathbf{R}$ . For a set  $\Sigma$  of tgd's the transformed set is  $\hat{\Sigma} = \{\hat{\xi} : \xi \in \Sigma\}$ . Using the enrichment notion the following was shown.

THEOREM 3.[11]  $\Sigma \in \text{CT}_{\forall\forall}^{\text{obl}}$  if and only if  $\hat{\Sigma} \in \text{CT}_{\forall\forall}^{\text{std}}$ .

To relate the termination of the semi-oblivious chase to the standard chase termination, we use a transformation similar to the enrichment. This transformation is called *semi-enrichment* and takes a tuple generating dependency  $\xi = \alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z})$  over a schema  $\mathbf{R}$  and converts it into the tgd  $\hat{\xi} = \alpha(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \beta(\bar{x}, \bar{z}), H(\bar{x})$ , where  $H$  is a new relational symbol which does not appear in  $\mathbf{R}$ . For a set  $\Sigma$  of tgd's defined on schema  $\mathbf{R}$ , the transformed set is  $\tilde{\Sigma} = \{\hat{\xi} : \xi \in \Sigma\}$ . Using the semi-enrichment notion, the standard and the semi-oblivious chase can be related as follows.

THEOREM 4.  $\Sigma \in \text{CT}_{\forall\forall}^{\text{sobl}}$  if and only if  $\tilde{\Sigma} \in \text{CT}_{\forall\forall}^{\text{std}}$ .

*Proof:* Similar to the proof of Theorem 3. ■

We now turn our attention to the core chase. Note the core chase is deterministic since all active triggers are fired in parallel, before taking the core of the result. Thus we have:

PROPOSITION 2.  $\text{CT}_{I,\forall}^{\text{core}} = \text{CT}_{I,\exists}^{\text{core}}$  and  $\text{CT}_{\forall\forall}^{\text{core}} = \text{CT}_{\forall\exists}^{\text{core}}$ .

It is well known that all here considered chase variations compute a (finite) *universal model* of  $I$  and  $\Sigma$  when they terminate [9, 6, 4, 17]. In [6], Deutsch et al. showed that if  $I \cup \Sigma$  has a universal model, the core chase will terminate in an instance that is the core of all universal models. We thus have

PROPOSITION 3.

1.  $\text{CT}_{I,\exists}^{\text{std}} \subset \text{CT}_{I,\forall}^{\text{core}}$ , for any instance  $I$ .
2.  $\text{CT}_{\forall\exists}^{\text{std}} \subset \text{CT}_{\forall\forall}^{\text{core}}$ .

*Proof:* (Sketch) To see that the inclusion in part 2 of the proposition is strict, let  $\Sigma = \{R(x) \rightarrow \exists z R(z)S(x)\}$ , and  $I_0 = \{R(a)\}$ . In this setting there will be exactly one active trigger at each step, and the algorithm will converge only at the infinite instance

$$\bigcup_{i>1} \{R(z_i), S(z_{i-1})\} \cup \{R(a), S(a), R(z_1)\}.$$

From this, it follows that  $\Sigma \notin \text{CT}_{\forall\forall}^{\text{std}}$  and  $\Sigma \notin \text{CT}_{\forall\exists}^{\text{std}}$ . Note that for any positive integer  $i$ , the core of the instance  $I_i$  is  $\{R(a), S(a)\}$ . Thus the core chase will terminate at instance  $I_1 = \{R(a), S(a)\}$ . ■

The following Corollary highlights the relationship between the termination classes.

COROLLARY 1.  $\text{CT}_{\forall\forall}^{\text{obl}} \subset \text{CT}_{\forall\forall}^{\text{sobl}} \subset \text{CT}_{\forall\forall}^{\text{std}} \subset \text{CT}_{\forall\exists}^{\text{std}} \subset \text{CT}_{\forall\forall}^{\text{core}}$ .

## 4.2 Undecidability of termination

It has been known for some time that “chase termination is undecidable.” Specifically, the following results have been obtained in the literature so far.

THEOREM 5.

1.  $\text{CT}_{I,\forall}^{\text{std}}$  and  $\text{CT}_{I,\exists}^{\text{std}}$  are RE-complete [6].
2.  $\text{CT}_{I,\forall}^{\text{core}} = \text{CT}_{I,\exists}^{\text{core}}$ , and both sets are RE-complete [6].
3.  $\text{CT}_{I,\forall}^{\text{sobl}} = \text{CT}_{I,\exists}^{\text{sobl}}$ , and both sets are RE-complete [17].
4. Let  $\Sigma$  be a set of guarded tgd's [4]. Then the question  $\Sigma \in \text{CT}_{I,\forall}^{\text{core}}$  is decidable [13].

Our aim is to provide a systematic overview on the complexity of all termination classes. We shall first show that  $\text{CT}_{\forall\forall}^{\text{core}}$  is coRE-complete. To achieve this, we provide a uniform reduction for both the  $\text{CT}_{I,\forall}^{\text{core}}$  and  $\text{CT}_{\forall\forall}^{\text{core}}$  problems, thus reproving Theorem 5, part 2 as a side effect. We also note that the proofs in [6] rely on machine reductions, and that Marnette observes in [18] that a proof using a higher level problem, such as Post's correspondence problem, is still lacking. We fill this gap using word-rewriting systems as redact. A word-rewriting system, also known as semi-Thue systems, is a set of rules of the form  $\ell \rightarrow r$ , where  $\ell$  and  $r$  are words over a finite alphabet  $\Delta$ . Let  $u$  and  $v$  be words in  $\Delta^*$ . The  $u$  can be *derived* from  $v$  if  $u = x\ell y$  and  $v = xry$ , for some  $x, y \in \Delta^*$  and rule  $\ell \rightarrow r$ . A word rewriting system is terminating for a word  $w$  if the derivation closure of  $w$  is finite. The system is uniformly terminating if it is terminating for all words  $w \in \Delta^*$ . We will prove that for every word rewriting system and word there is a set

of dependencies and an instance such that the rewriting system is terminating for that word if and only if the core chase with the corresponding dependencies on the corresponding instance terminates. We also prove that if the core chase with the corresponding dependencies is infinite on all instances, then the rewriting system is uniformly terminating. It has long been known that testing if a word-rewriting system terminates for a given input word is RE-complete [5], and that testing if a word-rewriting system is uniformly terminating is coRE-complete [15].

The next theorem states the main result of this section.

**THEOREM 6.** *The membership problem for  $\text{CT}_{\forall\forall}^{\text{core}}$  is coRE-complete.*

The proof involves a development using a series of lemmas, and can be found in its entirety in the Appendix. Our reduction from word rewriting systems also yields

**COROLLARY 2.** *The membership problem for  $\text{CT}_{I,\forall}^{\text{core}}$  is RE-complete (cf. [6]).*

The same reduction from word-rewriting systems gives the following undecidability result too:

**THEOREM 7.** *The membership problem for  $\text{CT}_{\forall\exists}^{\text{std}}$  is coRE-complete.*

Unfortunately the reduction used for the previous results can't be used to show the undecidability of the  $\text{CT}_{\forall\forall}^{\text{std}}$ ,  $\text{CT}_{\forall\forall}^{\text{sobl}}$  or  $\text{CT}_{\forall\forall}^{\text{obl}}$  classes. To overcome this we have allow a single *denial constraint*, that is a tgd of the form  $\alpha \rightarrow \perp$ , which is satisfied by an instance  $I$  only if there is no homomorphism  $h$ , such that  $h(\alpha) \notin I$ . It is an open problem if the following result (or part of it) can be obtained without such constraints.

**THEOREM 8.** *In case the set of dependencies may contain at least one denial constraint, the membership problems for  $\text{CT}_{\forall\forall}^{\text{std}}$ ,  $\text{CT}_{\forall\forall}^{\text{sobl}}$  and  $\text{CT}_{\forall\forall}^{\text{obl}}$  are coRE-complete.*

As a final observation we need to mention that the class “TOC” of mappings defined in [17], for which termination of the semi-oblivious class is proved to be RE-complete, is not the same with the class  $\text{CT}_{\forall\forall}^{\text{sobl}}$ . Also there is no direct reduction from TOC to  $\text{CT}_{\forall\forall}^{\text{sobl}}$  membership problem, as former is defined only for sets of tgd's describing *data exchange mappings*, and the question is whether the chase terminates for all instances that are source instances for the data exchange setting.

## 5. GUARANTEED TERMINATION

To overcome the undecidability of chase termination, a flurry restricted classes of tgd's have been proposed in the literature. These classes have been put forth as subsets of  $\text{CT}_{\forall\forall}^{\text{std}}$ , although at the time only  $\text{CT}_{I,\forall}^{\text{core}}$  was known to be undecidable. In this section we review these restricted classes with the purpose of determining their overall structure and termination properties.

Before reviewing these classes of sets of tgd's let us define two properties attached to such classes based on the enrichment and semi-enrichment rewritings defined in subsection 4.1.

A class of sets of tgd's  $\mathcal{C}$  is said to be *closed under enrichment* if  $\Sigma \in \mathcal{C}$  implies that  $\widehat{\Sigma} \in \mathcal{C}$ . Using this notation together with Theorem 3 gives us a sufficient condition for a class of dependencies to belong to  $\text{CT}_{\forall\forall}^{\text{obl}}$ :

**PROPOSITION 4.** *Let  $\mathcal{C} \subseteq \text{CT}_{\forall\forall}^{\text{std}}$  such that  $\mathcal{C}$  is closed under enrichment. Then  $\mathcal{C} \subseteq \text{CT}_{\forall\forall}^{\text{obl}}$ .*

Using this proposition we will reveal classes of dependencies that ensure termination for the oblivious chase. Similarly we define the notion of *semi-enrichment closure* for classes of dependency sets. The semi-enrichment closure property gives a sufficient condition for the semi-oblivious chase termination.

**PROPOSITION 5.** *Let  $\mathcal{C} \subseteq \text{CT}_{\forall\forall}^{\text{std}}$  such that  $\mathcal{C}$  is closed under semi-enrichment. Then  $\mathcal{C} \subseteq \text{CT}_{\forall\forall}^{\text{sobl}}$ .*

As we will see next, most of the known classes that ensure the standard chase termination are closed under semi-enrichment, and thus those classes actually guarantee the semi-oblivious chase termination as well. As we saw in Section 3, the semi-oblivious chase has a lower complexity than the standard chase.

### Acyclicity based classes

As full tgd's do not generate any new nulls during the chase, any sequence with a set of full tgd's will terminate since there only is a finite number of tuples that can be formed out of the elements of the domain of the initial instance. The cause of non-termination lies in the existentially quantified variables in the head of the dependencies. Most restricted classes thus rely on restricting the tgd's in a way that prevents these existential variables to participate in any recursion.

The class of *weakly acyclic* sets of tgd's [9] was one of the first restricted classes to be proposed. Consider

$$\Sigma_1 = \{R(x, y) \rightarrow \exists z S(z, x)\}.$$

Let  $(R, 1)$  denote the first position in  $R$ , and  $(S, 2)$  the second position in  $S$ , and so on. In a chase step based on this dependency the values from position  $(R, 1)$

get copied into the position  $(S, 2)$ , whereas the value in position  $(R, 1)$  “cause” the generation of a new null value in  $(S, 1)$ . This structure can be seen in the *dependency graph* of  $\Sigma_1$  that has a “copy” edge from vertex  $(R, 1)$  to vertex  $(S, 2)$ , and a “generate” edge from vertex  $(R, 1)$  to vertex  $(S, 1)$ . Note that the graph does not consider any edges from  $(R, 2)$  because variable  $y$  does not contribute to the generated values. The chase will terminate since there is no recursion going through the  $(S, 2)$  position. By contrast, the dependency graph of

$$\Sigma_2 = \{R(x, y) \rightarrow \exists z R(y, z)\}$$

has a generating edge from  $(R, 2)$  to  $(R, 2)$ . It is the generating self-loop at  $(R, 2)$  which causes the chase on for example the instance  $\{R(a, b)\}$  to converge only at the infinite instance  $\{R(a, b), R(b, z_1)\} \cup \{R(z_i, z_{i+1}) : i = 1, 2, \dots\}$ . The class of weakly acyclic tgds (WA) is defined to be those sets of tgd’s whose dependency graph doesn’t have any cycles involving a generating edge [9]. It is easy to observe that the class (WA) is closed under semi-enrichment but it is not closed under enrichment. This is because in the case of semi-enrichment the new relational symbol  $H$  considered for each dependency contains only variables that appears both in the body and the head of the dependency, and the new  $H$  atoms appear only in the heads of the semi-enriched dependency. This means that the dependency graph for a semi-enriched set of WA tgd’s will only add edges oriented into positions associated with the new relational symbol. The set  $\Sigma = \{R(x, y) \rightarrow \exists z R(x, z)\}$  shows that this is not the case for enrichment as  $\Sigma \in \text{WA}$  but  $\widehat{\Sigma} \notin \text{WA}$ .

The slightly smaller class of sets of tgd’s with *stratified witness* (SW) [7] was introduced around the same time as WA. An intermediate class, the *richly acyclic* tgd’s (RA) was introduced in [14] in a different context and it was later shown in [11] that  $\text{RA} \in \text{CT}_{\forall\forall}^{\text{obl}}$ . It can be easily verified that both classes SW and RA are closed under enrichment. The *safe dependencies* (SD [19]), and the *super-weakly acyclic* (sWA [17]) ones are both generalizations of the WA class, and both are close under semi-enrichment.

All of these classes have been proven to have PTIME membership tests, and have the following properties.

THEOREM 9. [7, 9, 19, 17, 11]

1.  $\text{SW} \subset \text{RA} \subset \text{WA} \subset \text{SD} \subset \text{sWA}$ .
2.  $\text{WA} \subset \text{CT}_{\forall\forall}^{\text{std}}$ ,  $\text{RA} \subset \text{CT}_{\forall\forall}^{\text{obl}}$ , and  $\text{sWA} \subset \text{CT}_{\forall\forall}^{\text{sobl}}$ .

In order to complete the picture suggested by the previous theorem we need a few more results. Consider

$$\Sigma_3 = \{R(x, y) \rightarrow \exists z R(x, z)\}.$$

Clearly  $\Sigma_3 \in \text{WA}$ . Let  $I_0 = \{R(a, b)\}$ , and consider a semi-oblivious chase sequence  $I_0, I_1, I_2, \dots$ . It is easy to see that for any  $I_i, i > 0$ , there exists a (non-active) trigger  $(\xi, \{x/a, y/z_i\})$ , meaning that the oblivious chase will not terminate. Thus we have  $\Sigma_3 \notin \text{CT}_{\forall\forall}^{\text{obl}}$ . On the other hand, for the set

$$\Sigma_4 = \{S(y), R(x, y) \rightarrow \exists z R(y, z)\},$$

we have  $\Sigma_4 \in \text{CT}_{\forall\forall}^{\text{obl}}$ . Furthermore,  $\Sigma_4 \notin \text{WA}$ , since the dependency graph of  $\Sigma_4$  will have a generating self-loop on vertex  $(R, 2)$ . This gives us

PROPOSITION 6. *The classes WA and  $\text{CT}_{\forall\forall}^{\text{obl}}$  are incomparable wrt inclusion.*

It was shown in [19] that  $\text{WA} \subset \text{SD}$  and also that  $\text{SD} \subset \text{CT}_{\forall\forall}^{\text{std}}$ . We can now extend this result by showing that, similarly to the WA class, the following holds:

PROPOSITION 7. *The classes SD and  $\text{CT}_{\forall\forall}^{\text{obl}}$  are incomparable wrt inclusion.*

*Proof:* (Sketch) The proof consists of showing that  $\Sigma_3 \in \text{SD} \setminus \text{CT}_{\forall\forall}^{\text{obl}}$ , and that  $\Sigma_5 \in \text{CT}_{\forall\forall}^{\text{obl}} \setminus \text{SD}$ , where

$$\Sigma_5 = \{R(x, x) \rightarrow \exists y R(x, y)\}.$$

Details are omitted. ■

From the semi-enrichment closure of the WA and SD classes and Proposition 5 we get the following result.

PROPOSITION 8.  $\text{WA} \in \text{CT}_{\forall\forall}^{\text{sobl}}$  and  $\text{SD} \in \text{CT}_{\forall\forall}^{\text{sobl}}$ . Furthermore, for any instance  $I$  and any  $\Sigma \in \text{SD}$  the semi-oblivious chase with  $\Sigma$  on  $I$  terminates in time polynomial in the size of  $I$ .

Note that the previous result follows directly also from a similar result for the class sWA [17]. Still, as shown by the following proposition, the super-weakly acyclic class does not include the class of dependencies that ensures termination for the oblivious chase variation, nor does the inclusion hold in the other direction.

PROPOSITION 9. *sWA and  $\text{CT}_{\forall\forall}^{\text{obl}}$  are incomparable wrt inclusion.*

*Proof:* (Sketch) We exhibit the super-weakly acyclic set  $\Sigma_3 = \{R(x, y) \rightarrow \exists z R(x, z)\}$ . It is clear  $\Sigma_3 \notin \text{CT}_{\forall\forall}^{\text{obl}}$ . For the converse, let

$$\Sigma_6 = \{S(x), R(x, y) \rightarrow \exists z R(y, z)\}.$$

Then  $\Sigma_6 \notin \text{sWA}$ , on the other hand it can be observed that the oblivious chase with  $\Sigma_6$  terminates on all instances. This is because tuples with new nulls cannot cause the dependency to fire, as these new nulls will never be present in relation  $S$ . ■



## Stratification based classes

Consider  $\Sigma_7 = \{\xi_1, \xi_2\}$ , where

$$\begin{aligned}\xi_1 &= R(x, x) \rightarrow \exists z S(x, z), \text{ and} \\ \xi_2 &= R(x, y), S(x, z) \rightarrow R(z, x).\end{aligned}$$

In the dependency graph of  $\Sigma_7$  we will have the cycle  $(R, 1) \rightsquigarrow (S, 2) \rightsquigarrow (R, 1)$ , and since  $(S, 2)$  is an existential position, the set  $\Sigma_7$  is not weakly acyclic. However, it is easy to see that  $\Sigma_7 \in \text{CT}_{\forall\forall}^{\text{std}}$ . It is also easily seen that if  $S$  is empty and  $R$  non-empty, then  $\xi_1$  will “cause”  $\xi_2$  to fire for every tuple in  $R$ . Let us denote this relationship by  $\xi_1 < \xi_2$ . On the other hand, there is no chase sequence in which a new null in  $(S, 2)$  can be propagated back to a tuple in  $R$  and to fire a trigger based on  $\xi_1$ , and thus create an infinite loop. We denote this with  $\xi_2 \not< \xi_1$ . In comparison, when chasing with

$$\Sigma_8 = \{R(x, y) \rightarrow \exists z R(z, x)\},$$

the new null  $z_i$  in  $(z_i, z_{i-1})$  will propagate into tuple  $(z_{i+1}, z_i)$ , in an infinite regress. If we denote the  $\text{tgd}$  in  $\Sigma_8$  with  $\xi$ , we conclude that  $\xi < \xi$ . A formal definition of the  $<$  relation is given in the next section.

The preceding observations led Deutsch et al. [6] to define the class of *stratified* dependencies by considering the *chase graph* of a set  $\Sigma$ , where the individual  $\text{tgd}$ 's in  $\Sigma$  are the vertexes and there is an edge from  $\xi_1$  to  $\xi_2$  when  $\xi_1 < \xi_2$ . A set  $\Sigma$  is then said to be stratified if the vertex-set of every cycle in the chase graph forms a weakly acyclic set. The class of all sets of stratified  $\text{tgd}$ 's is denoted  $\text{Str}$ . In the previous example,  $\Sigma_7 \in \text{Str}$ , and  $\Sigma_8 \notin \text{Str}$ .

In [19] Meier et al. observed that  $\text{Str} \not\subseteq \text{CT}_{\forall\forall}^{\text{std}}$  and that actually only  $\text{Str} \subset \text{CT}_{\forall\exists}^{\text{std}}$ , and came up with a corrected definition of  $<$ , which yielded the *corrected stratified* class  $\text{CStr}$  of  $\text{tgd}$ 's, for which they showed

THEOREM 10. [19]

$$\text{CStr} \subset \text{CT}_{\forall\forall}^{\text{std}}, \text{Str} \subset \text{CT}_{\forall\exists}^{\text{std}}, \text{ and } \text{CStr} \subset \text{Str}.$$

From the observation that the  $\text{CStr}$  class is closed under semi-enrichment and from Proposition 5 we have:

PROPOSITION 10.

1.  $\text{CStr} \subset \text{CT}_{\forall\forall}^{\text{sobl}}$ .
2.  $\text{CStr}$  and  $\text{CT}_{\forall\forall}^{\text{obl}}$  are incomparable wrt inclusion.

*Proof:* (Sketch) For the second part we have  $\Sigma_3 \in \text{CStr}$  and  $\Sigma_3 \notin \text{CT}_{\forall\forall}^{\text{obl}}$ . For the converse consider the dependency set  $\Sigma_6$  from the proof of Proposition 9. ■

Meier et al. [20] further observed that the basic stratification definition also catches some false negatives.

For this they considered dependency set  $\Sigma_9 = \{\xi_3, \xi_4\}$ , where

$$\begin{aligned}\xi_3 &= S(x), E(x, y) \rightarrow E(y, x), \text{ and} \\ \xi_4 &= S(x), E(x, y) \rightarrow \exists z E(y, z), E(z, x).\end{aligned}$$

Here  $\xi_3$  and  $\xi_4$  belong to the same stratum according to the definition of  $\text{CStr}$ . Since new nulls in both  $(E, 1)$  and  $(E, 2)$  can be caused by  $(E, 1)$  and  $(E, 2)$ , there will be generating self-loops on these vertexes in the dependency graph. Hence  $\Sigma_9 \notin \text{CStr}$ . On the other hand, it is easy to see that the number of new nulls that can be generated in the chase is bounded by the number of tuples in relation  $S$  in the initial instance. Consequently  $\Sigma_9 \in \text{CT}_{\forall\forall}^{\text{std}}$ .

In order to avoid such false negatives, Meier et al. [20] gave an alternative definition of the  $<$  relation and of the chase graph. Both of these definitions are however technically rather involved, and will not be repeated here. The new *inductively restricted* class, abbreviated  $\text{IR}$ , restricts each connected component in the modified chase graph to be in  $\text{SD}$ . In example above,  $\Sigma_9 \in \text{IR}$ .

Meier et al. [20] also observed that  $\text{IR}$  only catches *binary* relationships  $\xi_1 < \xi_2$ . This could be generalized to a ternary relation  $<(\xi_1, \xi_2, \xi_3)$ , meaning that there exists a chase sequence such that firing  $\xi_1$  will cause  $\xi_2$  to fire, and this in turn causes  $\xi_3$  to fire. This will eliminate those cases where  $\xi_1, \xi_2$  and  $\xi_3$  form a connected component in the (modified) chase graph, and yet there is no chase sequence that will fire  $\xi_1, \xi_2$  and  $\xi_3$  in this order. Thus the tree dependencies should not be in the same stratum.

Similarly to the ternary extension, the  $<$  relation can be generalized to be  $k$ -ary. The resulting termination classes are denoted  $\text{T}[k]$ . Thus  $\text{T}[2] = \text{IR}$ , and in general  $\text{T}[k] \subset \text{T}[k+1]$  [20]. The main property is

THEOREM 11. [19]

$$\text{CStr} \subset \text{IR} = \text{T}[2] \subset \text{T}[3] \subset \dots \subset \text{T}[k] \subset \dots \subset \text{CT}_{\forall\forall}^{\text{std}}.$$

To complete the picture, we have the following proposition based on the semi-oblivious closure for the  $\text{T}[k]$  hierarchy and Proposition 5.

PROPOSITION 11.

1.  $\text{T}[k] \subset \text{CT}_{\forall\forall}^{\text{sobl}}$ .
2.  $\text{T}[k]$  and  $\text{CT}_{\forall\forall}^{\text{obl}}$  are incomparable wrt inclusion.

Before concluding this section need to mention that all the classes discussed here are closed under semi-enrichment, thus they ensure the termination for the less expensive semi-oblivious chase in a polynomial number of steps, in the size of the input instance.

The Hasse diagram in Figure 2 from the Appendix summarizes the stratification based classes and their termination properties.

## 6. COMPLEXITY OF STRATIFICATION

As we noted in Section 5, all the acyclicity based classes have the property that testing whether a given set  $\Sigma$  belongs to it can be done in PTIME. The situation changes when we move to the stratified classes. The authors of [6] claimed that testing if  $\xi_1 < \xi_2$  is in NP for a given  $\xi_1$  and  $\xi_2$ , thus resulting in Str having a coNP membership problem. We shall see in Theorem 12 below that this cannot be the case, unless NP = coNP. We shall use the  $<$  order as it is defined for the CStr class. The results also hold for the Str class. First we need a formal definition.

**DEFINITION 1.** [19] *Let  $\xi_1$  and  $\xi_2$  be tgd's. Then  $\xi_1$  precedes  $\xi_2$ , denoted  $\xi_1 < \xi_2$ , if there exists an instance  $I$  and homomorphisms  $h_1$  and  $h_2$  from the universal variables in  $\xi_1$  and  $\xi_2$ , respectively, such that:*

- (i)  $I \models h_2(\xi_2)$ , and
- (ii)  $I \xrightarrow{(\xi_1, h_1)} J$  using an oblivious chase step, and
- (iii)  $J \not\models h_2(\xi_2)$ .

Note that the pair  $(\xi_1, h_1)$  in the previous definition denotes a trigger, not necessarily an active trigger, because the chase step considered is the oblivious one. Intuitively, the instance  $I$  in the definition is a witness to the “causal” relationship between  $\xi_1$  and  $\xi_2$  (via  $h_2$ ), as  $h_2(\xi_2)$  won't fire at  $I$ , but will fire once  $\xi_1$  has been applied. The notion of stratum of  $\Sigma$  is as before, i.e. we build a chase graph consisting of a vertex for each tgd in  $\Sigma$ , and an edge from  $\xi_1$  to  $\xi_2$  if  $\xi_1 < \xi_2$ . Then  $\xi_1$  and  $\xi_2$  are in the same *stratum* when they both belong to the same cycle in the chase graph of  $\Sigma$ . A set  $\Sigma$  of tgd's is said to be *C-stratified* (CStr) if all its strata are weakly acyclic [19].

**THEOREM 12.**

1. *Given two tgd's  $\xi_1$  and  $\xi_2$ , the problem of testing if  $\xi_1 < \xi_2$  is coNP-hard.*
2. *Given a set of dependencies  $\Sigma$ , the problem of testing if  $\Sigma \in \text{CStr}$  is NP-hard.*

**Proof:** For part 1 of the theorem we will use a reduction from the graph 3-colorability problem that is known to be NP-complete. It is also well known that a graph  $G$  is 3-colorable iff there is a homomorphism from  $G$  to  $K_3$ , where  $K_3$  is the complete graph with 3 vertices. We provide a reduction  $G \mapsto \{\xi_1, \xi_2\}$ , such that  $G$  is not 3-colorable if and only if  $\xi_1 < \xi_2$ .

We identify a graph  $G = (V, E)$ , where  $|V| = n$  and  $|E| = m$  with the sequence

$$G(x_1, \dots, x_n) = E(x_{i_1}, y_{i_1}), \dots, E(x_{i_m}, y_{i_m}),$$

and treat the elements in  $V$  as variables. Similarly, we identify the graph  $K_3$  with the sequence  $K_3(z_1, z_2, z_3) =$

$$E(z_1, z_2), E(z_2, z_1), E(z_1, z_3), E(z_3, z_1), E(z_2, z_3), E(z_3, z_2)$$

where  $z_1, z_2$ , and  $z_3$  are variables. With these notations, given a graph  $G = (V, E)$ , we construct tgd's  $\xi_1$  and  $\xi_2$  as follows:

$$\begin{aligned} \xi_1 &= R(z) \rightarrow \exists z_1, z_2, z_3 K_3(z_1, z_2, z_3), \text{ and} \\ \xi_2 &= E(x, y) \rightarrow \exists x_1, \dots, x_n G(x_1, \dots, x_n). \end{aligned}$$

Clearly the reduction is polynomial in the size of  $G$ . We will now show that  $\xi_1 < \xi_2$  iff  $G$  is not 3-colorable.

First, suppose that  $\xi_1 < \xi_2$ . Then there exists an instance  $I$  and homomorphisms  $h_1$  and  $h_2$ , such that  $I \models h_2(\xi_2)$ . Consider  $J$ , where  $I \xrightarrow{(\xi_1, h_1)} J$ . Thus  $R^I$  had to contain at least one tuple, and  $E^I$  had to be empty, because otherwise the monotonicity property of the chase we would imply that that  $J \models h_2(\xi_2)$ .

On the other hand, we have  $I \xrightarrow{(\xi_1, h_1)} J$ , where instance  $J = I \cup \{K_3(h'_1(z_1), h'_1(z_2), h'_1(z_3))\}$ , and  $h'_1$  is a distinct extension of  $h_1$ . Since  $E^I = \emptyset$ , and we assumed that  $J \not\models h_2(\xi_2)$ , it follows that there is no homomorphism from  $G$  into  $J$ , i.e. there is no homomorphism from  $G(h'_2(x_1), \dots, h'_2(x_n))$  to  $K_3(h'_1(z_1), h'_1(z_2), h'_1(z_3))$ , where  $h'_2$  is a distinct extension of  $h_2$ . Therefore the graph  $G$  is not 3-colorable.

For the other direction, let us suppose that graph  $G$  is not 3-colorable. This means that there is no homomorphism from  $G$  into  $K_3$ . With these assumption let us consider  $I = \{R(a)\}$  homomorphism  $h_1 = \{z/a\}$  and homomorphism  $h_2 = \{x/h'_1(z_1), y/h'_1(z_2)\}$ . It is easy to verify that  $I$ ,  $h_1$  and  $h_2$  satisfy the three conditions for  $\xi_1 < \xi_2$ .

For part 2 of the theorem, consider the set  $\Sigma = \{\xi_1, \xi_2\}$  defined as follows:

$$\begin{aligned} \xi_1 &= R(z_1, v) \rightarrow \exists z_2, z_3, w K_3(z_1, z_2, z_3), \\ &\quad R(z_2, w), R(z_3, w), S(w), \text{ and} \\ \xi_2 &= E(x, y) \rightarrow \exists x_1, \dots, x_n, v G(x_1, \dots, x_n), R(x, v). \end{aligned}$$

It is straightforward to verify that  $\Sigma \notin \text{WA}$  and that  $\xi_2 < \xi_1$ . Similarly to the proof of part 1, it can be shown that  $\xi_1 < \xi_2$  iff the graph  $G$  is not 3-colorable. From this follows that  $\Sigma \in \text{CStr}$  iff there is no cycle in the chase graph, iff  $\xi_1 \not< \xi_2$  iff  $G$  is 3-colorable. ■

Note that the reduction in the previous proof can be used to show that the problem  $\Sigma \in \text{Str}$  is NP-hard. Similar result can be also obtained for the IR class and also for the *local stratification* based classes introduced by Greco et al. in [12]. The obvious upper bound for the problem  $\xi_1 < \xi_2$  is given by:

PROPOSITION 12. *Given two dependencies  $\xi_1$  and  $\xi_2$ , the problem of determining whether  $\xi_1 < \xi_2$  is in  $\Sigma_2^P$ .*

**Proof:** From [6] we know that if  $\xi_1 < \xi_2$  there is an instance  $I$  satisfying Definition 1, such that size of  $I$  is bounded by a polynomial in the size of  $\{\xi_1, \xi_2\}$ . Thus, we can guess instance  $I$ , homomorphisms  $h_1$  and  $h_2$  in NP time. Next, with a NP oracle we can check if  $I \models h_2(\xi_2)$  and  $J \not\models h_2(\xi_2)$ , where  $I \xrightarrow{(\xi_1, h_1)} J$ . ■

We shall see that the upper bound of the proposition actually can be lowered to  $\Delta_2^P$ . For this we need the following characterization theorem.

THEOREM 13. *Let  $\xi_1 = \alpha_1 \rightarrow \beta_1$  and  $\xi_2 = \alpha_2 \rightarrow \beta_2$  be *tgds*. Then,  $\xi_1 < \xi_2$  if and only if there is an atom  $t$ , and homomorphisms  $h_1$  and  $h_2$ , such that the following hold.*

- (a)  $t \in \beta_1$ ,
- (b)  $h_1(t) \in h_2(\alpha_2)$ ,
- (c)  $h_1(t) \notin h_1(\alpha_1)$ , and
- (d) *There is no idempotent homomorphism from  $h_2(\beta_2)$  to  $h_2(\alpha_2) \cup h_1(\alpha_1) \cup h_1(\beta_1)$ .*

**Proof:** We first prove the “only if” direction. For this, suppose that  $\xi_1 < \xi_2$ , that is, there exists an instance  $I$  and homomorphisms  $g_1$  and  $g_2$ , such that conditions (i) – (iii) of Definition 1 are fulfilled.

From conditions ii and iii we have that  $g_1(\alpha_1) \subseteq I$  and  $g'_1(\beta_1) \not\subseteq I$ , for any distinct extension  $g'_1$  of  $g_1$ .

Now, consider  $h_1 = g_1$  and  $h_2 = g_2$ . Let  $t$  be an atom from  $\beta_1$  such that  $h'_1(t) \in h'_1(\beta_1) \cap h_2(\alpha_2)$  and  $h'_1(t) \notin h_1(\alpha_1)$ , for an extension  $h'_1$  of  $h_1$ . Such an atom  $t$  must exist, since otherwise it will be that  $h'_1(\beta_1) \cap h_2(\alpha_2) \subseteq h_1(\alpha_1)$ , which is not possible because of conditions (i) and (iii) (note that  $h_1 = g_1$ ). It is now easy to see that  $t$ ,  $h'_1$  and  $h_2$  satisfy conditions (a), (b), and (c) of the theorem. It remains to show that condition (d) also is satisfied. By construction we have  $J = I \cup h'_1(\beta_1)$ . It now follows that  $I \cup h'_1(\beta_1) \not\models h_2(\xi_2)$ . Because  $h_1(\alpha_1) \subseteq I$ , condition (d) is indeed satisfied.

For the “if” direction of the theorem, suppose that there exists an atom  $t$  and homomorphisms  $h_1$  and  $h_2$ , such that conditions (a), (b), (c) and (d) holds. Let  $g_1 = h_1$ ,  $g_2 = h_2$  and let  $I = (h_1(\alpha_1) \cup h_2(\alpha_2)) \setminus h'_1(t)$ , for a distinct extension  $h'_1$  of  $h_1$ . Because  $h'_1(t) \notin I$  and  $h'_1(t) \in h_2(\alpha_2)$ , it follows that  $h_2(\alpha_2) \not\subseteq I$ . Thus we have  $I \models h_2(\xi_2)$ , proving point (i) of Definition 1. On the other hand, because point (c) of the theorem is assumed, it follows that  $h_1(\alpha_1) \subseteq I$ , from which we get  $I \xrightarrow{(\xi_1, h_1)} J$ , where  $J = I \cup h'_1(\beta_1)$ , proving points (i) and (ii) from Definition 1. Since  $I \cup h'_1(\beta_1) = h_1(\alpha_1) \cup h_2(\alpha_2) \cup h'_1(\beta_1)$ , and point (d) holds, we get  $J \not\models h_2(\xi_2)$ ,

thus showing that condition (iii) of Definition 1 is also satisfied. ■

It is easy to note that by adding the extra condition (e) *there is no idempotent homomorphism from  $\beta_1$  to  $\alpha_1$*  in the previous theorem we obtain a characterization of the stratification order associated with the Str class.

With this characterization result we can now tighten the  $\Sigma_2^P$  upper bound of Proposition 12 as follows:

THEOREM 14. *Given two dependencies  $\xi_1$  and  $\xi_2$ , the problem of determining whether  $\xi_1 < \xi_2$  is in  $\Delta_2^P$ .*

**Proof:** For this proof we will use the characterization Theorem 13, and the observation that  $\Delta_2^P = \text{P}^{\text{coNP}}$ . Consider the following PTIME algorithm that enumerates all possible  $h_1, h_2$  and  $t$ :

```

1  for  $t \in \beta_1$ 
2      do for all  $(h_1, h_2) = \text{mgu}(t, \alpha_2)$ 
3          do if  $h_1(t) \notin h_1(\alpha_1)$  return  $t, h_1, h_2$ 

```

In the algorithm,  $\text{mgu}(t, \alpha_2)$  denotes all pairs  $(h_1, h_2)$  such that there exists an atom  $t' \in \alpha_2$ , with  $h_1(t) = h_2(t')$ , and there is no  $(g_1, g_2)$  and  $f$  different from the identity mappings, such that  $h_1 = g_1 \circ f$  and  $h_2 = g_2 \circ f$ . Using the values returned by previous algorithm and with a coNP oracle we can test if point (d) holds. Thus, the problem is in  $\Delta_2^P$ . ■

Armed with these results we can now state the upper-bound for the complexity of the CStr membership problem.

THEOREM 15. *Let  $\Sigma$  be a set of *tgds*, then the problem of testing if  $\Sigma \in \text{CStr}$  is in  $\Pi_2^P$ .*

**Proof:** (Sketch) To prove that  $\Sigma$  is not in CStr guess a set of tuples  $(\xi_1, t^1, h_1^1, h_2^1), \dots, (\xi_k, t^k, h_1^k, h_2^k)$ , where  $\xi_1, \dots, \xi_k$  are *tgds* in  $\Sigma$ ,  $t_1, \dots, t_k$  are atoms, and  $h_1^i, h_2^i$  are homomorphisms, for  $i \in \{1, \dots, k\}$ . Then, using an NP oracle check that  $\xi_i < \xi_{i+1}$ , for  $i \in \{1, \dots, k-1\}$ , and  $\xi_k < \xi_1$ , using the characterization Theorem 13 with  $t^i$ ,  $h_1^i$ ,  $h_2^i$  and  $t^k$ ,  $h_1^k$ ,  $h_2^k$  respectively. And then check in PTIME if the set of dependencies  $\{\xi_1, \dots, \xi_k\}$  is not weakly acyclic. Thus, the complexity is  $\Pi_2^P$ . ■

We note that using the obvious upper-bound  $\Sigma_2^P$  for testing if  $\xi_1 < \xi_2$ , the membership problem for the class CStr would be in  $\Pi_3^P$ . As mentioned the same results apply also for the class Str. Even if the complexity bounds for testing if  $\xi_1 < \xi_2$  are not tight, it can be noted that a coNP upper bound would not lower the  $\Pi_2^P$  upper bound of the membership problem for CStr.

## 7. CONCLUSIONS

We have undertaken a systematization of the somewhat heterogeneous area of the chase. Our analysis produced a taxonomy of the various chase versions and their termination properties, showing that the main sufficient classes that guarantee termination for the standard chase also ensures termination for the complexity-wise less expensive semi-oblivious chase. Even if the standard chase procedure in general captures more sets of dependencies that ensure the chase termination than the semi-oblivious chase we argue that for most practical constraints the semi-oblivious chase is a better choice. We have also proved that the membership problem for the classes  $CT_{VV}^{core}$  and  $CT_{V\exists}^{std}$  is **coRE**-complete and in case we allow also at least one denial constraint the same holds for  $CT_{VV}^{std}$ ,  $CT_{VV}^{sobl}$  and  $CT_{VV}^{obl}$ . Still it remains an if the membership problem for  $CT_{VV}^{std}$  remains **coRE**-complete without denial constraints. The same also holds for the classes  $CT_{VV}^{sobl}$  and  $CT_{VV}^{obl}$ . Finally we have analyzed the complexity of the membership problem for the class of stratified sets of dependencies. Our bounds for this class are not tight, and it remains an open problem to pinpoint the complexity exactly.

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## APPENDIX

Note that the theorems presented in the paper we kept the same numbering and the new theorems/lemmas are numbered continuously.

### 4. Chase termination questions

THEOREM 2.

$$\text{CT}_{\forall\forall}^{\text{obl}} = \text{CT}_{\forall\exists}^{\text{obl}} \subset \text{CT}_{\forall\forall}^{\text{sobl}} = \text{CT}_{\forall\exists}^{\text{sobl}} \subset \text{CT}_{\forall\forall}^{\text{std}} \subset \text{CT}_{\forall\exists}^{\text{std}}.$$

**Proof:** (Sketch) We will only show the strict inclusion parts of the theorem. For the first inclusion, let  $\Sigma = \{R(x, y) \rightarrow \exists z R(x, z)\}$ . It is easy to see that  $\Sigma \in \text{CT}_{\forall\forall}^{\text{sobl}}$  and  $\Sigma \notin \text{CT}_{\forall\exists}^{\text{sobl}}$ . The second strict inclusion  $\text{CT}_{\forall\forall}^{\text{sobl}} \subset \text{CT}_{\forall\forall}^{\text{std}}$  is more intricate, as most sets of dependencies in  $\text{CT}_{\forall\forall}^{\text{std}}$  part of  $\text{CT}_{\forall\forall}^{\text{sobl}}$ . To distinguish the two classes, let  $\Sigma = \{\xi_1, \xi_2\}$ , where

$$\begin{aligned}\xi_1 &= R(x) \rightarrow \exists z S(z), T(z, x), \text{ and} \\ \xi_2 &= S(x) \rightarrow \exists z' R(z'), T(x, z').\end{aligned}$$

Let now  $I$  be an arbitrary instance, and suppose that  $I = \{R(a_1), \dots, R(a_n), S(b_1), \dots, S(b_m)\}$ . There is no loss of generality, since if the standard chase with  $\Sigma$  on  $I$  terminates, then it will also terminate even if the initial instance contains atoms over the relational symbol  $T$ . It is easy to see that all standard chase sequences with  $\Sigma$  on  $I$  will terminate in the instance  $A \cup B$ , where

$$\begin{aligned}A &= \bigcup_{i \in \{1, \dots, n\}} \{R(a_i), S(z_i), T(z_i, a_i)\}, \text{ and} \\ B &= \bigcup_{i \in \{1, \dots, m\}} \{R(z'_i), S(b_i), T(b_i, z'_i)\}.\end{aligned}$$

On the other hand, all semi-oblivious chase sequences will converge only at the infinite instance

$$\bigcup_{k \in \mathbb{N}} (C_k \cup D_k) \cup A \cup B,$$

where

$$\begin{aligned}C_k &= \bigcup_{i=1}^m \{S(z_{kn+(k-1)m+i}), T(z_{kn+(k-1)m+i}, z'_{(k-1)m+(k-1)n+i})\} \cup \\ &\quad \bigcup_{j=1}^n \{S(z_{kn+km+j}), T(z_{kn+km+j}, z'_{km+(k-1)n+j})\}, \\ D_k &= \bigcup_{i=1}^n \{R(z'_{km+(k-1)n+i}), T(z_{(k-1)n+(k-1)m+i}, z'_{km+(k-1)n+i})\} \cup \\ &\quad \bigcup_{j=1}^m \{R(z'_{km+kn+j}), T(z_{kn+(k-1)m+j}, z'_{km+kn+j})\}.\end{aligned}$$

For the last inclusion  $\text{CT}_{\forall\forall}^{\text{std}} \subset \text{CT}_{\forall\exists}^{\text{std}}$ , consider the set  $\Sigma = \{R(x, y) \rightarrow R(y, y); R(x, y) \rightarrow \exists z R(y, z)\}$ . Clearly  $\Sigma \in \text{CT}_{\forall\exists}^{\text{std}}$ , because for any instance all chase sequences that start by firing the first tgd will terminate. On the other hand  $\Sigma \notin \text{CT}_{\forall\forall}^{\text{std}}$ , as the standard chase

with  $\Sigma$  does not terminate on  $I = \{R(a, b)\}$  whenever the second tgd is applied first. ■

### 4.2 Undecidability of termination

Hitherto, the following results have been obtained.

THEOREM 5.

1.  $\text{CT}_{I, \forall}^{\text{std}}$  and  $\text{CT}_{I, \exists}^{\text{std}}$  are RE-complete [6].
2.  $\text{CT}_{I, \forall}^{\text{core}} = \text{CT}_{I, \exists}^{\text{core}}$ , and both sets are RE-complete [6].
3.  $\text{CT}_{I, \forall}^{\text{sobl}} = \text{CT}_{I, \exists}^{\text{sobl}}$ , and both sets are RE-complete [17].
4. Let  $\Sigma$  be a set of guarded tgd's [4]. Then the question  $\Sigma \in \text{CT}_{I, \forall}^{\text{core}}$  is decidable [13].

Before describing the reduction and the proofs for theorems 6, 7 and 8 let us first give a brief description for the word-rewriting systems.

**Word rewriting systems.** Let  $\Delta$  be a finite set of symbols, denoted  $a, b, \dots$ , possibly subscripted, and  $\Delta^*$  the set of all finite words over  $\Delta$ . Let  $\Theta$  be a finite subset of  $\Delta^* \times \Delta^*$ . Treating each pair in  $\Theta$  as a *rule*, the relation  $\Theta$  gives rise to a *rewriting relation*  $\rightarrow_{\Theta} \subseteq \Delta^* \times \Delta^*$  defined as  $\{(u, v) : u = xly, v = xry, (\ell, r) \in \Theta\}$ .

We use the notation  $u \rightarrow_{\Theta} v$  instead of  $\rightarrow_{\Theta}(u, v)$ . If  $\Theta$  is understood from the context we will simply write  $u \rightarrow v$ . If we want to emphasize which rule  $\rho \in \Theta$  was used we write  $u \rightarrow_{\rho} v$ . When  $u \rightarrow_{\rho} v$  we say that  $v$  is obtained from  $u$  by a *rewriting step* (based on  $\rho$ ). By  $u \rightarrow^n v$  we mean that  $v$  can be obtained from  $u$  in at most  $n$  rewriting steps. A *rewriting system* is then a pair  $(\Delta^*, \Theta)$ . If  $\Delta$  is understood from the context we shall denote a rewriting system simply with  $\Theta$ .

A sequence  $w_0, w_1, w_2, \dots$  of words from  $\Delta^*$  is said to be a  $\Theta$ -*derivation sequence* (or simply a derivation sequence), if  $w_i \rightarrow w_{i+1}$  for all  $i = 0, 1, 2, \dots$ . A derivation sequence might be finite or infinite. The *termination problem for  $\Theta$*  and a word  $w_0 \in \Delta^*$ , is to determine whether all derivation sequences  $w_0, w_1, w_2, \dots$  originating from  $w_0$  are finite. The *uniform termination problem for  $\Theta$*  is to determine whether for *all* words  $w_0 \in \Delta^*$ , it holds that all derivation sequences  $w_0, w_1, w_2, \dots$  originating from  $w_0$  are finite. It has long been known that the termination problem is RE-complete [5], and that the uniform termination problem is coRE-complete [15].

We now describe our reduction  $\Theta \mapsto \Sigma_{\Theta}$ . We assume without loss of generality that  $\Delta = \{0, 1\}$ . The tgd set  $\Sigma_{\Theta}$  is defined for schema  $\mathbf{R}_{\Theta} = \{E, E^*, L, R, D\}$  and consists of  $\{\xi_{\rho} : \rho \in \Theta\} \cup \{\xi_{L_0}, \xi_{L_1}, \xi_{R_0}, \xi_{R_1}\} \cup TC \cup AD \cup S$ , where  $\xi_{\rho}$  is:

$$\begin{aligned}E(x_0, a_1, x_1), \dots, E(x_{n-1}, a_n, x_n) &\rightarrow \exists y_0, \dots, y_m L(x_0, y_0), \\ E(y_0, b_1, y_1), \dots, E(y_{m-1}, b_m, y_m), R(x_n, y_m).\end{aligned}$$

when  $\rho = (a_1 \dots a_n, b_1 \dots b_m)$ . We will also have the following “grid creation” rules, using left  $L$  and right  $R$  predicates.

$$\begin{aligned}
\xi_{L_0} &= E(x_0, 0, x_1), L(x_1, y_1) \rightarrow \exists y_0 L(x_0, y_0), E(y_0, 0, y_1) \\
\xi_{L_1} &= E(x_0, 1, x_1), L(x_1, y_1) \rightarrow \exists y_0 L(x_0, y_0), E(x_0, 1, y_1) \\
\xi_{R_0} &= R(x_0, z_0), E(x_0, 0, x_1) \rightarrow \exists z_1 E(z_0, 0, z_1), R(x_1, z_1) \\
\xi_{R_1} &= R(x_0, z_0), E(x_0, 1, x_1) \rightarrow \exists z_1 E(z_0, 1, z_1), R(x_1, z_1)
\end{aligned}$$

In the sequel we will assume, unless otherwise stated, that all instances are over schema  $\mathbf{R}_\Theta$ . For an instance  $I$  the following rule set  $AD$  (“active domain”) computes  $\text{dom}(I) \cup \Delta$  in relation  $D$ .

$$\begin{aligned}
&\rightarrow D(0), D(1) \\
E(x, z, y) &\rightarrow D(x), D(z), D(y) \\
L(x, y) &\rightarrow D(x), D(y) \\
R(x, y) &\rightarrow D(x), D(y) \\
E^*(x, y) &\rightarrow D(x), D(y)
\end{aligned}$$

Given an instance  $I$  we denote by  $G_I$  the graph with edge set:

$$\{(x, y) : E(x, z, y) \in I \text{ or } E^*(x, y) \in I \text{ or } L(x, y) \in I \text{ or } R(x, y) \in I\}.$$

The following set  $TC$  computes in  $E^*$  the transitive closure of  $G_I$ :

$$\begin{aligned}
E(x, z, y) &\rightarrow E^*(x, y) \\
L(x, y) &\rightarrow E^*(x, y) \\
R(x, y) &\rightarrow E^*(x, y) \\
E^*(x, y), E^*(y, z) &\rightarrow E^*(x, z)
\end{aligned}$$

If an instance has a cycle in  $E^*$ , that is there exists a cycle in  $G_I$ , the dependencies in the following “saturation” set  $S$  will be fired:

$$\begin{aligned}
E^*(v, v), D(x), D(z), D(y) &\rightarrow E(x, z, y) \\
E^*(v, v), D(x), D(y) &\rightarrow L(x, y), R(x, y), E^*(x, y)
\end{aligned}$$

In the sequel, we shall sometimes say that an instance  $I$  is cyclic (acyclic) if  $G_I$  is cyclic (acyclic). We shall also speak of “the graph of  $I$ ”, when we mean  $G_I$ .

We denote by  $H_I$  the Herbrand base of instance  $I$ , i.e. the instance where, for each relation symbol  $R \in \mathbf{R}_\Theta$ , the interpretation  $R^{H_I}$  contains all tuples (of appropriate arity) that can be formed from the constants in  $(\text{dom}(I) \cap \text{Cons}) \cup \Delta$ . The proof of the following lemma is straightforward:

LEMMA 1. *core(I) = H<sub>I</sub>, whenever H<sub>I</sub> is a subinstance of I.*

LEMMA 2. *Let I be an arbitrary instance over schema  $\mathbf{R}_\Theta$ , and let  $I = I_0, I_1, I_2, \dots$  be the core chase sequence with  $\Sigma_\Theta$  on I. If there is an integer i and a constant or variable x, such that  $E^*(x, x) \in I_i$  (i.e. the graph  $G_{I_j}$  is cyclic for some  $j \leq i$ ), then the core chase sequence is finite.*

*Proof:* (Sketch) First we note that  $H_{I_k} = H_I$  for any instance  $I_k$  in the core chase sequence, since the chase does not add any new constants. If the core chase does not terminate at the instance  $I_i$  mentioned in the claim, it follows that the dependencies in the set  $S$  will fire at  $I_i$  and generate  $H_I$  as a subinstance. It then follows from Lemma 1 that  $I_{i+1} = H_I$ . It is easy to see that  $H_I \models \Sigma_\Theta$ , so the core chase will terminate at instance  $I_{i+1}$ . ■

Intuitively the previous lemma guarantees that whenever we have a cycle in the initial instance the core chase process will terminate. Thus, in the following we will not have to care about instances that may contain cycles.

The following lemma ensures that the core chase with  $\Sigma_\Theta$  on an acyclic instance will not create any cycles.

LEMMA 3. *Let I be an arbitrary acyclic instance over schema  $\mathbf{R}_\Theta$ , and let  $I = I_0, I_1, I_2, \dots$  be the core chase sequence with  $\Sigma_\Theta$  on I. Then  $G_{I_i}$  is acyclic, for all instances  $I_i$  in the sequence.*

*Proof:* (Sketch) Suppose to the contrary that  $G_{I_i}$  is cyclic, for some  $I_i$  in the sequence. Wlog we assume that  $I_i$  is the first such instance in the sequence. Clearly  $i \geq 1$ . This means that by applying all active triggers on  $I_{i-1}$  will add a cycle (note that the taking the core cannot add a cycle). Let  $(\xi_1, h_1), \dots, (\xi_n, h_n)$  be the triggers that add tuples to  $I_i$ , causing  $G_{I_i}$  to be cyclic. First, it is easy to see that  $\{\xi_{L_0}, \xi_{L_1}, \xi_{R_0}, \xi_{R_1}\} \cap \{\xi_1, \dots, \xi_n\} = \emptyset$ . This is because these dependencies do not introduce any new edges in  $G_{I_i}$  between vertices in  $G_{I_{i-1}}$ , they only add a new vertex into  $G_{I_i}$  which will have two incoming edges from vertices already in  $G_{I_{i-1}}$ . A similar reasoning shows that none of the  $\xi_\rho \in TC$  or  $\xi \in AD$  can be part of the set  $\{\xi_1, \dots, \xi_n\}$ . Finally, the dependencies in the set  $S$  may introduce cycles and may thus be part of the set  $\{\xi_1, \dots, \xi_n\}$ . But the dependencies in  $S$  are fired only when  $E^*(x, x) \in I_{i-1}$ , which means that  $G_{I_{i-1}}$  already contains a cycle, namely the self-loop on  $x$ . Contradicts our counter assumption that  $I_i$  is the first instance in the chase sequence that contains a cycle. ■

We still need a few more notions. A *path*  $\pi$  of an instance  $I$  over  $R_\Theta$  is a set

$$\{E(x_0, a_1, x_1), E(x_1, a_2, x_2), \dots, E(x_{n-1}, a_n, x_n)\}$$

of atoms of  $I$ , such that  $\{a_1, a_2, \dots, a_n\} \subseteq \Delta$  (recall that  $\Delta$  is the alphabet of the rewriting system  $\Theta$ ). The word

spelled by the path  $\pi$  is

$$\text{word}(\pi) = a_1 a_2 \dots a_n.$$

A *max-path*  $\pi$  in an instance  $I$  is a path, such that no path  $\pi'$  in  $I$  is a strict superset of  $\pi$ . We can now relate words and instances as follows: let  $I$  be an acyclic instance, we define

$$\text{paths}(I) = \{\pi : \pi \text{ is a max-path in } I\}.$$

Clearly  $\text{paths}(I)$  is finite, for any finite instance  $I$ . Conversely, let  $w = a_1 a_2 \dots a_n \in \Delta^*$ . We define

$$I_w = \{E(x_0, a_1, x_1), E(x_1, a_2, x_2), \dots, E(x_{n-1}, a_n, x_n)\},$$

where the  $x_i$ 's are pairwise distinct variables. Clearly  $\text{paths}(I_w) = \{\pi\}$ , where  $\text{word}(\pi) = w$ .

LEMMA 4. *Let  $w \in \Delta^*$ , and let  $I_w = I_0, I_1, I_2, \dots$  be the core chase sequence with  $\Sigma_\Theta$  on  $I_w$ . For each instance  $I_i$  in the sequence, denote by  $I'_i$  the instance obtained from  $I_i$  by firing all active triggers, that is  $I_{i+1} = \text{core}(I'_i)$ . Then  $\text{paths}(I'_i) = \text{paths}(I_{i+1})$ .*

*Proof:* (Sketch) If there were a path  $\pi$ , such that  $\pi \in \text{paths}(I'_i) \setminus \text{paths}(\text{core}(I'_i))$ , there would have to be atoms of the form  $L(x, x)$  and  $R(x, x)$  in the instance  $I'_i$ , which would contradict Lemma 3 as  $I_w$ , by definition, does not contain any cycles. ■

In order to be able to relate rewrite sequences and core chase sequences we introduce rewrite trees and path trees. For a rewriting system  $\Theta$  and word  $w \in \Delta^*$ , we construct the *rewrite tree*  $\mathcal{T}_w$  inductively. Start with a root node labelled  $w$ . Then, for each leaf node  $n$  in  $\mathcal{T}_w$ , for each possible derivation  $v \rightarrow u$ , where  $v$  is the label of  $n$ , add a new node  $m$  as a child of  $n$ , and label  $m$  with  $u$ .

The next lemma follows directly from the construction of  $\mathcal{T}_w$ .

LEMMA 5. *Let  $w \in \Delta^*$ . Then  $\mathcal{T}_w$  has an infinite branch if and only if there is an infinite rewriting derivation  $w = w_0, w_1, w_2, \dots$  generated by  $\Theta$  from  $w$ .*

We next define the *path tree*  $\mathcal{P}_w$  of the core chase sequence  $I_w = I_0, I_1, I_2, \dots$  generated by  $\Sigma_\Theta$  from  $I_w$ . The path tree is defined inductively on the levels of the tree. First, let  $\mathcal{P}_w$  consist of a single node, labelled with the single path in  $\text{paths}(I_w)$ . Inductively, for each leaf node  $n$  in  $\mathcal{P}_w$ , where  $\pi$  is the label of  $n$ ,

1. for each  $\xi_\rho = \alpha \rightarrow \beta \in \Sigma_\Theta$  if there is a homomorphism  $h$ , such that the trigger  $(\xi_\rho, h)$  is active on  $I_i$  and  $h(\alpha) \subseteq \pi$ , then add a child  $m$  labelled with the unique path in  $\text{paths}(h'(\beta))$ , where  $h'$  is the distinct extension of  $h$ .

2. for each pair  $(\xi_L, \xi_R)$ , say  $(\xi_{L_0}, \xi_{R_1})$ , if there is a homomorphism  $h$ , such that  $(\xi_{L_0}, h)$  is active on  $I_i$  (and/or  $(\xi_{R_1}, h)$  is active on  $I_i$ ), and  $\pi$  contains  $E(h(x_0), 0, h(x_1))$  (and/or  $E(h(x_0), 1, h(x_1)) \in \pi$ ), let  $h'$  be the distinct extension of  $h$ , and add a child node  $m$ , labelled with the unique path in the set

$$\text{paths}(E(h'(y_0), 0, h'(y_1)) \cup \pi)$$

(labelled with the unique path in

$$\text{paths}(E(h'(y_0), 0, h'(y_1)) \cup \pi \cup E(h(x_0), 1, h(z_1)))$$

or with the unique path in

$$\text{paths}(\pi \cup E(h'(z_0), 1, h'(z_1))),$$

respectively).

Similarly to Lemma 5, we have

LEMMA 6. *Let  $w \in \Delta^*$ . Then  $\mathcal{P}_w$  is infinite if and only if the core chase sequence  $I_w = I_0, I_1, I_2, \dots$  on  $I_w$  with  $\Sigma_\Theta$  is infinite.*

We can now state the following important theorem.

THEOREM 16. *Let  $w \in \Delta^*$ . Then the core chase sequence  $I_w = I_0, I_1, I_2, \dots$  with  $\Sigma_\Theta$  on  $I_w$  is infinite if and only if there is an infinite derivation  $w = w_0, w_1, w_2, \dots$  generated by  $\Theta$ .*

*Proof:* (Sketch) For the if part, suppose that there is an infinite derivation  $w = w_0, w_1, w_2, \dots$ . From Lemma 5 it follows that  $\mathcal{T}_w$  has an infinite branch and by construction the branch is labelled by the derivation sequence  $w = w_0, w_1, w_2, \dots$  generated by  $\Theta$ . We claim that there is an infinite branch in  $\mathcal{P}_w$  labelled with  $\pi_0, \pi_1, \pi_2, \dots$ , and a sequence of indices  $0 = j_0 < j_1 < j_2 < \dots$ , such that  $\text{word}(\pi_{j_i}) = w_i$ , for all  $i = 0, 1, 2, \dots$ . Clearly  $\text{word}(\pi_0) = w_0$ . For the inductive hypothesis fix  $n$ , and suppose that  $\text{word}(\pi_{j_i}) = w_i$ , for all  $i = 0, 1, \dots, n$ . Let  $w_n = xly$  and  $w_{n+1} = xry$ . Also, let  $k = \max\{|x|, |y|\}$ . It follows from the inductive hypothesis and the construction of  $\mathcal{P}_w$ , that the branch labelled  $\pi_{j_0}, \dots, \pi_{j_1}, \dots, \pi_{j_n}$ , where  $\text{word}(\pi_{j_i}) = w_i$ , continues with nodes labelled  $\pi_{j_n+1}, \pi_{j_n+2}, \dots, \pi_{j_n+k}$ , where  $\text{word}(\pi_{j_n+k}) = w_{n+1}$ . Since  $\mathcal{P}_w$  thus has an infinite branch, Lemma 6 tells us that the core chase sequence is infinite as well. Figure 1 shows the relationship between  $\mathcal{P}_w$  and  $\mathcal{T}_w$  where  $\Theta = \{(0, 1)\}$  and the initial word  $w_0$  is 1101.

For the other direction, suppose that the core chase sequence is infinite. From Lemma 6 it follows that  $\mathcal{P}_w$  has an infinite branch. Let this branch be labelled  $\pi_0, \pi_1, \pi_2, \dots$ . We claim that there is an infinite derivation  $w = w_0, w_1, w_2, \dots$  generated by  $\Theta$ , and a sequence  $0 = j_0 < j_1 < j_2 < \dots$  of indices, such that  $\text{word}(\pi_{j_i}) = w_i$ , for all  $i = 0, 1, 2, \dots$ . This can be seen by choosing

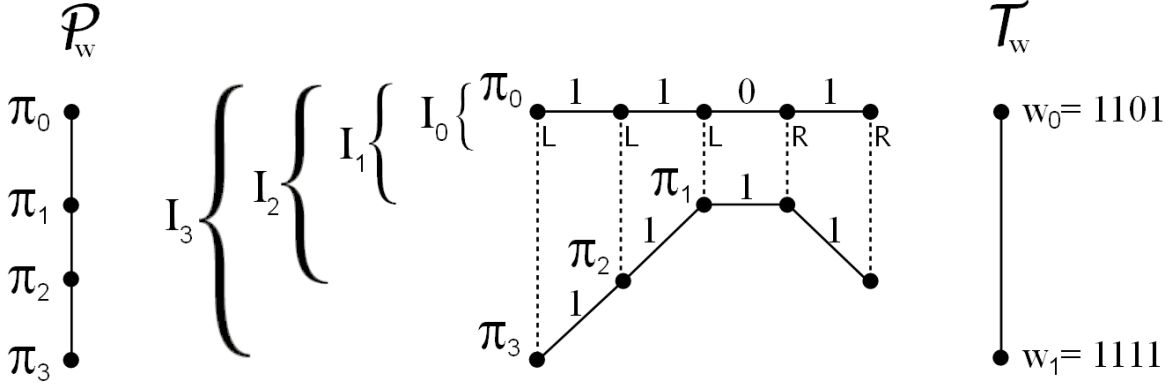


Figure 1: A simple reduction and the corresponding trees.

$j_{i+1} = j_i + k$ , where  $\pi_{j_i+k}$  is the first label on the path in  $\mathcal{P}_w$ , such that  $\pi_{j_i} \notin \pi_{j_i+k}$ . ■

This theorem together with the RE-completeness result for rewriting termination [5], yields the undecidability result of Deutsch et al. [6] for core chase termination on a given instance.

**COROLLARY 3.** *The set  $\text{CT}_{I,\forall}^{\text{core}}$  is RE-complete.*

Next we shall relate the uniform termination problem with the set  $\text{CT}_{\forall}^{\text{core}}$ . This means that we need to consider arbitrary instances, not just instances of the form  $I_w$ , for  $w \in \Delta^*$ . First, we introduce a few more notations.

We denote by  $\Sigma_\rho$  the set of all  $\xi_\rho$  dependencies in  $\Sigma_\Theta$ . Likewise,  $\Sigma_{LR}$  will denote the set  $\{\xi_{L_0}, \xi_{L_1}, \xi_{R_0}, \xi_{R_1}\}$ .

Let  $I$  be an acyclic instance, and  $I' = \text{Chase}_{\Sigma_{LR}}^{\text{core}}(I)$ . Where  $\text{Chase}_{\Sigma}^{\text{core}}(I)$  represents the instance returned by the core chase procedure on  $I$  with  $\Sigma$ . It is easy to see that  $I'$  is finite for any finite acyclic instance  $I$ . Then let  $I^* = \bigcup \{I_{\text{word}(\pi)} : \pi \in \text{paths}(I')\}$ . Intuitively,  $I^*$  is obtained from  $I$  by taking each max-path in  $I'$ , and making it into a unique line tree in  $G_{I^*}$ .

**LEMMA 7.** *Let  $I$  be an arbitrary acyclic instance. Then, the core chase of  $I$  with  $\Sigma_\Theta$  terminates if and only if the core chase of  $\text{Chase}_{\Sigma_{LR}}(I)$  with  $\Sigma_\Theta$  terminates.*

*Proof:* Let  $I = I_0, I_1, \dots$  be the core chase sequence of  $I$  with  $\Sigma_\Theta$  and  $\text{Chase}_{\Sigma_{LR}}(I) = J_0, J_1, \dots$  be the core chase sequence of  $\text{Chase}_{\Sigma_{LR}}(I)$  with  $\Sigma_\Theta$ . Let us first suppose that the core chase sequence on  $\text{Chase}_{\Sigma_{LR}}(I)$  does not terminate. In this case it is easy to see that there must be an integer  $i$  such that  $\text{Chase}_{\Sigma_{LR}}(I) \subseteq I_i$ , but from this follows that the core chase for  $I$  with  $\Sigma_\Theta$  does not terminate either. The other direction follows directly from the observation that for any  $i$  we have  $I_i \subseteq J_i$ . ■

**LEMMA 8.** *Let  $I$  be an arbitrary acyclic instance such that  $I \models \Sigma_{LR}$ . Then the core chase of  $I$  with  $\Sigma_\Theta$  is*

*infinite if and only if the core chase of  $I^*$  with  $\Sigma_\Theta$  is infinite.*

*Proof:* (Sketch) Let  $I = I_0, I_1, I_2, \dots$  be the core chase sequence of  $I$  with  $\Sigma_\Theta$ . And let  $I^* = J_0, J_1, \dots$  be the core chase sequence of  $I^*$  with  $\Sigma_\Theta$ . Suppose that the sequence  $I_0, I_1, I_2, \dots$  is infinite.

We will prove by induction that for each  $i$  there exists a  $j$  such that for each path  $\pi \in \text{paths}(I_i)$  there exists a unique path  $\pi' \in \text{paths}(J_j)$  and  $\text{word}(\pi)$  is a factor of  $\text{word}(\pi')$ . This proving the if part of the lemma.

For the base case, let  $i = 0$  and consider  $j = 0$ . By the definition  $I^*$  contains all the paths in  $\text{paths}(I)$ . For the inductive step let us suppose that for a fixed  $i$  it holds that for any integer  $k \leq i$  there exists  $j_k$  such that for each path  $\pi \in \text{paths}(I_k)$  there exists a unique path  $\pi' \in \text{paths}(J_{j_k})$  such that  $\text{word}(\pi)$  is a factor of  $\text{word}(\pi')$ . For each path  $\pi \in \text{paths}(I_{i+1}) \setminus \text{paths}(I_i)$  we will assign a unique path  $\pi' \in \text{paths}(J_j) \setminus \text{paths}(J_{j_i})$  for some  $j \geq j_i$  by considering the following 2 cases:

*Case 1.*  $\pi$  was created by applying a  $\xi_\rho$  dependency for some  $\rho = (a_1 \dots a_n, b_1 \dots b_m) \in \Theta$ . In this case it needs to be that there exists  $\pi_0 \in \text{paths}(I_i)$  such that  $a_1 \dots a_n$  is a factor of  $\text{word}(\pi_0)$ . From the induction hypothesis it follows that there exists a  $j$  and a unique  $\pi'_0 \in \text{paths}(J_j)$  such that  $\text{word}(\pi_0)$  is a factor of  $\text{word}(\pi'_0)$ . By transitivity of word factor relation it follows that  $a_1 \dots a_n$  is also a factor of  $\text{word}(\pi'_0)$ . But this means that the same dependency  $\xi_\rho$  can be applied, or was already applied, for  $I_j$ , following that  $\text{paths}(I_{j+1})$  contains the path  $\pi'$  such that  $\text{word}(\pi) = b_1 \dots b_m$  is a factor of  $\text{word}(\pi')$ .

*Case 2.*  $\pi$  was created by extending path  $\pi_1 \in \text{paths}(I_i)$  using one or two dependencies from  $\Sigma_{LR}$ . Because of the assumption  $I \models \Sigma_{LR}$  it follows that there must exist a subpath  $\pi_2$  of  $\pi_1$  (note that such a subpath is unique) such that  $\pi_2$  was obtained from a  $\xi_\rho$  dependency applied to path  $\pi_3 \in \text{paths}(I_k)$ , where  $k < i$  and  $\rho = (a_1 \dots a_n, b_1 \dots b_m)$ . Thus,  $\text{word}(\pi_2) = b_1 \dots b_m$  and



$word(\pi_3) = v_0 a_1 \dots a_n v_1$ , for some words  $v_0$  and  $v_1$ . Because the core computation does not shrink existing paths it follows that there must be a path  $\pi_4 \in paths(I_i)$  such that  $\pi_3$  is a subpath of  $\pi_4$ , thus the spelling for  $\pi_4$  will be  $word(\pi_4) = u_0 v_0 a_1 \dots a_n v_1 u_1$ , for some words  $u_0$  and  $u_1$ . Based on the acyclic requirement for  $I$ , Lemma 3, the definition for the  $\Sigma_{LR}$  dependencies and the observation that applying a  $\xi_\rho$  dependencies will always create a new path it follows that  $word(\pi)$  is a factor of the word  $u_0 v_0 b_1 \dots b_m v_1 u_1$ . From the induction hypothesis we have that for  $word(\pi_4) = u_0 v_0 a_1 \dots a_n v_1 u_1$  there exists a  $j$  and a unique path  $\pi' \in paths(J_j)$  such that  $word(\pi_4)$  is a factor of  $word(\pi')$ . This means that the core chase process can applied (or already applied) trigger  $(\xi_\rho, h)$  that mapped the body of  $\xi_\rho$  to the path given by word  $a_1 \dots a_n$  in  $word(\pi_4) = u_0 v_0 a_1 \dots a_n v_1 u_1$ . By applying this trigger instance  $J_{j+1}$  will contain the path  $\pi'_1$  that is a factor of  $u_0 v_0 b_1 \dots b_m v_1 u_1$ . After applying a maximum of  $\max(|u_0 v_0|, |v_1 u_1|)$  core chase steps (i.e. applying the copying dependencies) it follows that instance  $J_{j+1+\max(|u_0 v_0|, |v_1 u_1|)}$  will contain a path  $\pi'$  such that  $u_0 v_0 b_1 \dots b_m v_1 u_1$  is a factor of  $word(\pi')$ . Note that the assignment of  $\pi$  to  $\pi'$  is unique due to the uniqueness of the trigger applied. Also by the transitivity of the factor relation it follows that  $word(\pi)$  is also a factor of  $word(\pi')$ .

For the only if direction let us suppose that the core chase of  $I^*$  with  $\Sigma_\Theta$  does not terminate. Because all the paths in  $paths(I^*)$  does not share any node in common, it follows that there must be a path  $\pi$  in  $paths(I^*)$  such that the core chase of  $I_{word(\pi)}$  does not terminate. Clearly from the definition of  $I^*$  it follows that in the core chase sequence  $I = I_0, I_1, I_2, \dots$  there exists an integer  $i$  and there exists path  $\pi' \in paths(I_i)$  with  $word(\pi') = word(\pi)$ . This means that for the instance corresponding with the path  $\pi'$  the core chase will follow core chase steps isomorphic with the once used when chasing instance  $I_w$ , where  $w = word(\pi)$ . Thus, the core chase for  $\pi'$  will not terminate either. ■

We can now state the following important result

**THEOREM 17.** *A reduction system  $\Theta$  uniformly terminates if and only if the core chase terminates on all instances for  $\Sigma_\Theta$  (i.e.  $\Sigma_\Theta \in CT_{VV}^{core}$ ).*

*Proof:* (Sketch) First let us suppose that  $\Sigma_\Theta \in CT_{VV}^{core}$ . Let  $w \in \Delta^*$  be an arbitrary word. Because  $\Sigma_\Theta \in CT_{VV}^{core}$  it follows that the core chase will terminate also with instance  $I_w$ . From this and Theorem 16 it follows that the rewriting system  $\Theta$  will terminate for  $w$ .

For the other direction suppose that  $\Sigma_\Theta \notin CT_{VV}^{core}$ . Then there exists an instance  $I$ , such that the core chase sequence  $I = I_0, I_1, I_2, \dots$  with  $\Sigma_\Theta$  is infinite. From Lemma 2 it must be that  $I$  is acyclic. From Lemma 7 and Lemma 8 it follows that the core chase of  $I^*$  with  $\Sigma_\Theta$  must be infinite as well. But then there must be

path  $\pi \in I^*$  such that  $\Theta$  admits an infinite derivation starting from  $word(\pi)$ . But this means that  $\Theta$  is not uniformly terminating. ■

Using the previous result and the coRE-completeness result of Huet and Lankford [15], we now have the main theorem.

**THEOREM 6.** *The membership problem for  $CT_{VV}^{core}$  is coRE-complete.*

Up to here we proved the undecidability of the  $CT_{VV}^{core}$  class. Next we will show that this result can be extended with some minor changes to other termination classes as well.

**THEOREM 7.** *The membership problem for  $CT_{V\exists}^{std}$  is coRE-complete.*

*Proof:* (Sketch) It is easy to see that the same  $\Sigma_\Theta$  reduction works for the  $CT_{V\exists}^{std}$  case as well by choosing the branch that first applies all the  $AD \cup TC \cup S$  dependencies. This is because in case the initial arbitrary instance contains a cycle the full dependencies  $AD \cup TC \cup S$  will saturate the instance and the standard chase will terminate. If  $I$  does not contain any cycles then, as we showed, during the chase process no cycles are added and the termination proof is the same as for the core chase. ■

To show that the basic  $\Sigma_\Theta$  reduction can't be used for the  $CT_{VV}^{std}$  class. Consider the word-reduction system  $\Theta = \{(1, 0)\}$  and instance  $I = \{E(a, 0, a), L(a, b)\}$ . It is easy to see that the branch that applies the  $\xi_{L_0}$  dependency first will not terminate as it will generate the following infinite set of tuples:

$E$			$L$	
a	0	a	a	b
$x_1$	0	b	a	$x_1$
$x_2$	0	$x_1$	a	$x_2$
$\dots$			$\dots$	
$x_n$	0	$x_{n-1}$	a	$x_n$
$\dots$			$\dots$	

On the other hand, it is clear that the reduction system is uniformly terminating.

The undecidability result can still be obtained for the  $CT_{VV}^{std}$  class if we allow denial constraints. Then we simply define  $\Sigma_\Theta^\perp = \{E^*(x, x) \rightarrow \perp\} \cup (\Sigma_\Theta \setminus S)$ .

**THEOREM 8.** *Let  $\Sigma$  be a set of tgds and one denial constraint. The membership problem  $\Sigma \in CT_{VV}^{std}$  is coRE-complete.*

*Proof:* (Sketch) Similarly to the proof of Theorem 7 it is easy to see that if an arbitrary instance  $I$  contains a cycle, then the standard chase on  $I$  with  $\Sigma_\Theta^\perp$  will terminate on all branches. This is because the fairness

conditions guarantees that the denial constraint will be fired, and the chase will terminate. ■

Finally, we note that using the same  $\Sigma_{\Theta}^{\perp}$  reduction can be shown that the classes  $CT_{\forall\forall}^{sobl}$  and  $CT_{\forall\forall}^{obl}$  are also coRE-complete.

## 5. Guaranteed Termination

The following Hasse diagram summarizes the stratification based classes and their termination properties.

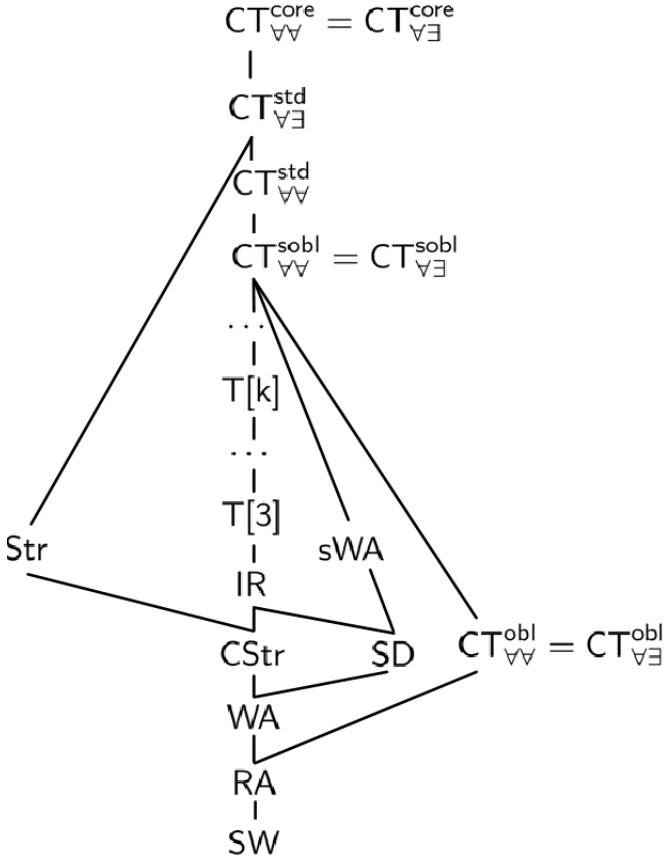


Figure 2: Sufficient classes.